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# NATIONAL MATHEMATICS MAGAZINE

VOL. XIX BATON ROUGE, LA., May, 1945 No. 8

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# A Grateful Acknowledgment

For three successive years kindly and generous help from the Mathematical Association of America has assisted this journal over rough places. A donation for the current year came in the form of a check forwarded by Secretary Walter B. Carver. Our apppreciation is deep, and sincerely do we trust that the Association does not object to this public announcement of their generosity.

S. T. Sanders.

# Evaluation of Real Roots by Means of Lower Degree Equations

By GLENN JAMES
*University of California at Los Angeles*

There are so many methods of approximating the roots of equations that one hesitates to propose another; but the following method seems to have enough advantages over current methods to justify its consideration. It contains almost none of the "trial by error" that makes Horner's Method tedious, and evaluates a root both from above and below, thus providing an upper bound to the error at each step, a feature needed in Newton's Method. In certain aspects it is closely related to the latter, as will appear in the context.

We first develop the method for algebraic equations then show that the method is valid for transcendental equations.

It is desired to find a real root, if such exist, of the equation

$$\text{(A)} \qquad f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \quad n > 2.$$

Let $h$ be an estimate to such a root and consider the following $n-1$ *associated* equations whose degrees are respectively,

$$n-1,\ n-2, \cdots,\ 1.$$

The first two equations are

$$f_1(x,h) = f(x) + \frac{h-x}{n} f'(x) = 0$$

and

$$f_2(x,h) = f_1(x,h) + \frac{h-x}{n-1} f_1'(x,h) = 0,$$

and in general

$$f_m(x,h) = f_{m-1}(x,h) + \frac{h-x}{n-m+1} f'_{m-1}(x,h) = 0,\ m = 1,2, \cdots, n-1,$$

and $f_0(x,h) = f(x)$.

These equations can be put in the following forms by substituting the first in the second, the second in the third, etc., and simplifying the last two that result, in a manner that will be explained in a moment:

$$(I_1) \qquad f_1(x,h) = f(x) + \frac{h-x}{n} f'(x) = 0$$

$$(I_2) \qquad f_2(x,h)=f(x)+2\frac{h-x}{n}f'(x)+\frac{(h-x)^2}{n(n-1)}f''(x)=0$$

$$(I_3) \qquad f_3(x,h)=f(x)+3\frac{h-x}{n}f'(x)+3\frac{(h-x)^2}{n(n-1)}f''(x)$$
$$+\frac{(h-x)^3f'''(x)}{n(n-1)(n-2)}=0$$

$$(I_m) \qquad f_m(x,h)=f(x)+\frac{m}{n}(h-x)f'(x)$$
$$+\frac{m(m-1)}{2!n(n-1)}(h-x)^2f''(x)+\cdots$$
$$+\frac{m(m-1)\cdots(m-r+2)}{(r-1)!n(n-1)\cdots(n-r+2)}(h-x)^{r-1}f^{[r-1]}(x)+\cdots$$
$$+\frac{(h-x)^mf^{[m]}(x)}{n(n-1)\cdots(n-m+1)}=0$$

$$(I_{n-2}) \quad f_{n-2}(x,h)=f(x)+\frac{n-2}{n}(h-x)f'(x)+\frac{(n-2)(n-3)}{2!n(n-1)}(h-x)^2f''(x)$$
$$+\cdots+\frac{(h-x)^{n-2}}{n(n-1)\cdots4\cdot3}f^{[n-2]}(x)$$
$$=f(h)+2\frac{(x-h)}{n}f'(h)+\frac{(x-h)^2}{n(n-1)}f''(h)=0$$

$$(I_{n-1}) \quad f_{n-1}(x,h)=f(x)+\frac{n-1}{n}(h-x)f'(x)+\frac{n-2}{2!n}(h-x)^2f''(x)$$
$$+\frac{n-3}{3!n}(h-x)^3f'''(x)+\cdots+\frac{(h-x)^{n-1}}{n!}f^{[n-1]}(x)$$
$$=f(h)+\frac{(x-h)}{n}f'(h)=0.$$

To aid in simplifying the second member of $(I_{n-2})$ and $(I_{n-1})$ we form also

$$(I_n) \qquad f_n(x,h)=f(x)+(h-x)f'(x)+\frac{(h-x)^2}{2!}f''(x)+\cdots$$
$$+\frac{(h-x)^{n-1}}{(n-1)!}f^{[n-1]}(x)+\frac{(h-x)^n}{n!}f^{[n]}(x)=f(h),$$

The last equality follows from the fact that $f_n(x,h)$ is of zero degree, i. e., a constant, and passes through the point $[h,f(h)]$ (or, of course, we could use Taylor's Theorem).

The second member of $(I_{n-1})$ can be written

$$f(x)+(h-x)f'(x)+\frac{(h-x)^2}{2!}f''(x)+\cdots+\frac{(h-x)^n}{n!}f^{[n]}(x)$$

$$-\frac{(h-x)}{n}\left[f'(x)+(h-x)f''(x)+\frac{(h-x)^2}{2!}f'''(x)+\cdots\right.$$

$$\left.+\frac{(h-x)^{n-1}}{(n-1)!}f^{[n]}(x)\right].$$

The quantity in the brackets is equal to $f'(h)$ by $(I_n)$. Hence this entire expression can be written.

$$f(h)+\frac{x-h}{n}f'(h).$$

The simplification of $f_{n-2}(x,h)$ is made similarly, the second member of $(I_{n-2})$ being separated into three polynomials to each of which $(I_n)$ is applied.

Similar simplifications can be made for $(I_{n-3})$, $(I_{n-4})$ etc., In fact, it can be proved by induction that

$$f_{n-h}(x,h)=f_h(h,x) \qquad h=\begin{cases}1,2,\cdots n/2, & n \text{ even}\\ 1,2,\cdots (n-1)/2, & n \text{ odd.}\end{cases}$$

However our prime interest here is in the quadratic and linear forms, since these are always solvable.

Suppose now that in some one of the associated equations $x$ has the values $R_1, R_2, R_3,\cdots$ corresponding to some set of values of $h$ which will be denoted by $h_1,h_2,h_3,\cdots$. It is then obvious that if $(R_i-h_i)$ approaches zero, as $i$ increases, $R_i$ approaches a root of $(A)$, and, of course, $h_i$ approaches the same root. We now define the $h_i$ in such a manner that the converse of this statement is true.

*Definition of the sequence* $[h_i]$. If $R_i$ is a root of

$$f_m(x,h_i)=0, \quad m=1,2,3,\cdots n-1,$$

then $$h_{i+1}=[(n-m)R_i+mh_i]/n.$$

For instance, in the case of the linear, or $(n-1)$th associated equation,

$$h_{i+1}=[R_i+(n-1)h_i]/n$$

and in the case of the first associated equation

$$h_{i+1}=[(n-1)R_i+h_i]/n.$$

This seems to be a very effective definition of the $h_i$ since the roots of the higher degree associated equations are generally nearer to the desired root of $(A)$ than are those of lower degree, as can be seen from the proof of Theorem III.

When using an associated equation of the 2nd or higher degree one would, of course, select for each $R_i$ that one of its roots which is nearest $h_i$.

The following basic theorem can now be stated:

*Theorem I.* A necessary and sufficient condition that $\lim_{i\to\infty} R_i^{[m]}$ be a root of $f(x)=0$ is that $[h_i-R_i^{[m]}]$, where $f_m(R_i^{[m]},h_i)=0$ and $h_{i+1}=[(n-m)R_i^{[m]}+mh_i]/n$, be a null sequence.

Before going into a rigorous discussion concerning the existence of a satisfactory $h_i$, we state our rule for solving algebraic equations and illustrate it.

*Rule for Approximating Real Roots.* Choose an associated equation, say $f_m(x,h)=0$ (which would usually be either the linear* or quadratic equation). Select an $h_i$ near a desired root of the given equation. Solve the equation

$$f_m(x,h_1)=0$$

for its root $R_1$ nearest $h_1$. Calculate

$$h_2=[(n-m)R_1+mh_1]/n.$$

Repeat this process $i$ times where $i$ is taken large enough to make

$$|h_{i+1}-R_i|/2$$

less than the numerical error allowable in the root. Then take for the root

$$(h_{i+1}+R_i)/2.$$

Find the real root of

$$x^3-2x-5=0$$

both by means of the linear and the quadratic associated equations. The former, according to $(I_{n-1})$, is

$$(3h^2-2)x-(4h+15)=0$$

*If $f'(h_i)=0$, the linear equation has no solution. In such a case use the quadratic. (See Theorem II).

and the latter, according to $(I_{n-2})$, is

$$3hx^2-4x-(15+2h)=0.$$

We carry the two methods in parallel columns, taking $h_1=2\cdot1$ in both cases.

| *Linear Case* | *Quadratic Case* |
|---|---|
| $R_1=(4h_1+15)/(3h_1^2-2)$ | $R_1=(2+\sqrt{4+45h+6h^2})/(3h)$ |
| $=2.0837+$ | $=2.09183$ |
| $h_2=(R_1+2h_1)/3$ | $h_2=(2R_1+h_1)/3$ |
| $=2.09457-$ | $=2.0945533$ |
| $R_2=2.09451844$ | $R_2=2.09455055$ |
| $h_2=2.09455180$ | $h_2=2.09455148$ |
| $(h_2-R_2)/2=0.00001668$ | $(h_2-R_2)/2=0.00000046$ |
| $R_3=2.0945514803$ | $(h_4-R_2)/2=0.00000000025$ |
| $h_4=2.0945514808$ | $(h_4+R_2)/2=\text{Root}=2.09455148055$ |

Thus two steps give a much better result with the quadratic; while three steps with the linear give much better than two with the quadratic.

(Incidently the above rule serves to find imaginary roots but the calculations are tedious and the choice of an $h_i$ is by no means as simple as it is in the case of real roots. For instance if one uses the linear associated equation to find an imaginary root of a real equation, he must certainly choose $h_1$ imaginary. For example a root of $x^2-x+1=0$ can be approximated (to three decimals in both the real and the imaginary parts) in three steps, by using $h_i=\sqrt{-1}$, but no root can be approximated by taking $h_1$ real.)

There is obviously considerable liberty in the choice of $h_1$ since the linear associated equation always has a solution unless $f'(h)=0$. However higher degree associated equations may fail to have solutions even when $f'(h)\neq0$. Such cases are taken care of by the following theorem:

*Theorem II.* If the equation

$$\varphi(x)+\frac{(h-x)}{m}\varphi'(x)=0 \tag{1}$$

has no solution, then all the roots of the equation $\varphi(x)=0$ are of the form

$$x=h+\sqrt[m]{k}. \tag{2}$$

Proof: If the given equation has no solution then it must be true that

$$\varphi(x)+\frac{h-x}{m}\varphi'(x) \quad \text{is a constant.} \tag{3}$$

Hence we seek the value of $y$ which satisfies the linear differential equation

$$y+\frac{h+x}{m}y'=k. \tag{4}$$

The solution of this differential equation is

$$y=c(x-h)^m+k, \tag{5}$$

as may be verified by substitution or derived by the classic method of elementary differential equations.

In the light of this theorem, when a selected associated equation reduces to a constant we use the next higher degree associated equation, the solution of which is simple.

An interesting elementary illustration of the use of this theorem is provided by using it to solve the quadratic

$$x^2+px+q=0. \tag{6}$$

Putting $m=2$ and $\varphi(x)=x^2+px+q$ in (3) gives

$$\left\{\frac{p}{2}+h\right\}x+\frac{hp}{2}+q$$

In order to make this expression a constant, $h$ must equal $-(p/2)$. Then by (5) we have

$$y=c\left\{x+\frac{p}{2}\right\}^2-\frac{p^2}{8}+q=0.$$

Comparison with (6) shows that $c=1$. Whence

$$x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}.$$

For real roots we have the following basic theorem:

*Theorem III.* If $r$ is a non-repeated real root of $f(x)=0$, there exist infinitely many $h_1$'s on both sides of $r$ such that

$$f_m(x,h_1)=0, \quad m=1,2,3,\cdots, n-1,$$

each has a root, $R_1^{[m]}$, on the side of $r$ opposite to $h_1$ and each of these roots taken with $h_1$ determines a sequence $[R_i^{[m]}]$ and a sequence $[h_i]$, where $f_m(R_i^{[m]},h_i)=0$ and $h_{i+1}=[(n-m)R_i+mh_i]/n$, which bound $r$ and approach it monotonically for $i>1$.

In proving this theorem, it suffices to discuss only the case in which $h_1>r$ and $f(x)$ increases in a neighborhood of $r$ including $h_1$. The other three possible cases are reflections of this one in the $x$-axis, and of these two in the line $x=r$.

Since $h_1-x$ and $f'(x)$ are both positive in the region under consideration, we have

$$f(x)<f_1(x,h)<f_2(x,h)<\cdots<f_{n-1}(x,h_1) \tag{7}$$

when $h_1-x$ is sufficiently small. For

$$f_m(x,h_1)-f_{m-1}(x,h_1)=1/n(h_1-x)f'(x)+1/n(h_1-x)^2\Theta(x)$$

where $\Theta(x)$ is a polynomial in $(h_1-x)$.

Now if $R_1'$, $R_1''$, $\cdots$ $R_1^{[m]}$ are respectively the roots of

$$f_1(x,h_1)=0, \quad f_2(x,h_1)=0, \cdots f_{n-1}(x,h_1)=0,$$

it follows from (7) that

$$r>R_1'>R_1''>\cdots>R_1^{[n-1]}. \tag{8}$$

Since $h_1>r$, any mean, between $h_1$ and any $R_1$, is less than $h_1$. Hence $h_2<h_1$.

If $y=f(x)$ is convex toward $x=h_1$, then $h_2>r$, as we shall now prove. From

$$h_2=[(n-m)R_1^{[m]}+mh_1]/n \tag{9}$$

and

$$f_{m-1}(R_1^{[m]},h_1)+\frac{h_1-R_1^{[m]}}{n-m+1}f'_{m-1}(R_1^{[m]},h_1)=0 \tag{10}$$

we have

$$h_2=R_1^{[m]}-\frac{m(n-m+1)}{n}f_{m-1}(R_1^{[m]},h_1)/f'_{m-1}(R_1^{[m]},h_1). \tag{11}$$

The right member is positive, for

$$f_{m-1}(R_1^{[m]},h_1)<0,$$

$$[m(n-m+1)/n>0$$

since $m<n$, and

$$f'_{m-1}(R_1^{[m]},h_1)=\frac{n-m}{n}f'(x)+(h_1-x)\varphi(x)>0,$$

for $|h_1-x|$ sufficiently small, since $f'(x)>0$ and $\varphi(x)$ is bounded. Hence $h_2$ is greater than $R_1^{[m]}$ plus the numerical value of the subtangent at the point whose abscissa is $R_1^{[m]}$; consequently it is greater than $r$. A similar argument holds for $h_3$ relative to $h_2$, etc. Thus we have proved that $h_i$ monotonically decreases toward $r$ when $y=f(x)$ is convex toward $y=h_1$ and $|h_1-x|$ is sufficiently small.

When $y=f(x)$ is concave toward $x=h_1$, $h_2<r$. For if we put $m=1$ in (11) and note from $(I_1)$ that $f_0(R_1',h_1)$ is $f(R_1')$, we have

$$h_2=R_1'-f(R_1')/f'(R_1')$$

which is obviously less than $r$ when $y=f(x)$ is concave toward $x=h_1$ in a neighborhood of $r$ including $R_1'$. The same argument holds for any associated equation relative to the next higher associated equation, because $py=f_m(x,h)$ are all convex in the same sense as $y=f(x)$ since

$$f''_m(x,h=\frac{(n-m)^2-(n-m)}{n(n-1)}f''(x)+(h-x)\psi(x).$$

Thus when it happens that $y=f(x)$ is concave toward $x=h_1,h_2$ falls on the convex side of this curve and serves as a new first estimate in one of the cases obtained by reflection, as mentioned above.

It remains to prove that, for a fixed $m,R_i^{[m]}$, $i=1,2,3,\cdots$ is monotonically increasing. It suffices to show that for $x$ fixed

$$f_m(,h_{i+1})<f_m(x,h_i),$$

or that $f_m(x,h)$ is an increasing function of $h$. From $(I_m)$

$$\partial f_m(x,h)/\partial h=\frac{m}{n}f'(x)+(h-x)\tau(x).$$

Thus $\partial f_m(x,h)/\partial h$ is positive for $|h-x|$ sufficiently small since $f'(x)$ is positive by hypothesis and $\tau(x)$ is bounded.

Although we have made frequent use of the notion of being "sufficiently small", in actual practice it is difficult to choose an $h_1$, which will not lead to some root by the use of most any associated equation. This is due to the fact that the graphs of all these equations pass through the point whose abscissa is $h_1$ and each of them passes through the bend points of the one of next higher degree, the first one passing through the bend points of $y=f(x)$.

*Application to the Solution of Transcendental Equations.* Theorems I and II are not dependent on $f(x)$ being a polynomial; and $n$ in equations $(I)$ may be looked upon as an arbitrary parameter, denoting the degree of $f(x)$ when we so desire. The simplifications of $(I_{n-2})$ and $(I_{n-1})$ are not altered if $f(x)$ is transcendental provided it possesses $n+1$ derivatives and the remainder which then occurs in $(I_n)$ is neglected. Finally $\Theta(x), \varphi(x), \psi(x)$ and $\tau(x)$ in the proof of Theorem III are bounded on $(x,h)$ if $f(x)$ and its first $n+2$ derivatives are bounded. Thus we see that the methods of this paper apply to ordinary transcendental equations. Consider the use of the linear associated equation

$$x = h - nf(h)/f'(h), \quad f'(h) \neq 0,$$

in finding a root in the neighborhood of which $f''(x) > 0$ and $f(x)$ is increasing. For $n=2$ an $h_1$ can be so chosen that it and the resulting $R_1$ bound $\tau$. Because if $h_1 > \tau$, then $h_1 - f(h_1)/f'(h_1) - \tau$ is positive and is an infinitesimal of higher order than $f(h_1)/f'(h_1)$.

*For example,* we approximate the positive root of

$$x^2 - \cos x = 0.$$

Since this root is obviously between 0 and 1, we choose $h_1 = 1$. Then

$$R_1 = h - f(h)/f'(h) = (h \sin h + 2 \cos h)/(2h + \sin h)$$
$$= .68, \quad h_1 = 1.$$
$$h_2 = (R_1 + h_1)/2 = .84$$
$$R_2 = .810$$
$$h_3 = .825$$
$$R_3 = .82324$$
$$h_4 = .82412$$
$$(h_4 + R_3)/2 = .82368$$

which we take for the root. The maximum numerical error in this root is

$$(h_4 - R_3)/2 = .00044.$$

*Comments.* This method gives the "Mechanics Rule" for extraction of square roots, when applied to the solution of

$$x^2 - a = 0, \quad a > 0.$$

For $$R_1 = h_1 - (h_1^2 - a)/h_1 = a/h_1$$

and $$h_2 = (a/h_1 + h_1)/2,$$

When $(I_1)$ is applied to $x^n-a=0$ it gives an approximation to the $n$th root in terms of the $(n-1)$th root. Here

$$R_1=(a/h_1)^{1/(n-1)}$$

and

$$h_2=[(n-1)(a/h_1)^{1/(n-1)}+h_1]/n.$$

The linear associated equation gives the Newtonian approximations for $h_i$ and these values minus $(n-1)$ times the subtangents at the points corresponding to $h_i$ for the $R_i$ sequence; and the first associated equation gives the Newtonian approximations each based on the previous $R_i$. For in the former case

$$R_i=h_i-nf(h_i)/f'(h_i)$$

and

$$h_{i+1}=[(n-1)h_i+R_i]/n=h_i-f(h_i)/f'(h_i)$$

while in the latter

$$h_i=R_i-nf(R_i)/f'(R_i)$$

and

$$h_{i+1}=(h_i+(n-1)R_i)/n=R_i-f(R_i)/f'(R_i).$$

Either one of these results follows from the other since

$$f_{n-1}(x,h)=f_1(h,x).$$

Analogously the other means, $h_i$, $i>1$, may be looked upon as generalizations of Newtonian approximations based on roots of associated equations.

# Applications of an Operator to Algebra and to Number-Theory, with Comments on the Tarry-Escott Problem

By HOWARD D. GROSSMAN
*New York City*

1. *The Operator D and its Functions.* The operator $D$ is defined by the equation $Df(x)=f'(x)$; its powers are defined by $D^{n+1}f(x)=D(D^nf(x))$, $(n=1, 2, 3, \cdots)$; and it obeys algebraic laws, particularly the formal equation,

$$e^D=1+D+D^2/2+\cdots .$$

Symbols other than $D$ have been found to be powerful instruments in summing complicated series, finite and infinite (Chrystal, Algebra, II, p. 405). The symbol $D$ may be used to prove very easily certain identities in algebra and in number-theory.

In the discussion that follows we take $f$ and $g$ to be polynomials. The following properties of functions of $D$ will be used:

1.1 The equation $f(x+h)=f(x)+hDf(x)+\frac{1}{2}h^2D^2f(x)+\cdots$

$$+\frac{h^nD^nf(x)}{n!}+\cdots$$

may be written symbolically $f(x+h)=(e^{hD})f(x)$.

1.2 If $f$ is of degree less than $n$, then $D^nf(x)=0$.

If $f$ is of degree $n$ with leading coefficient $c$, then $D^nf(x)=n!c$.

1.3 In a product of $n$ factors of the type $1-e^{hD}$, the lowest power of $D$ that occurs in the expansion is $n$ ( and its coefficient is $(-1)^n\Pi h$, since

$$1-e^{hD}=1-(1+hD+h^2D^2/2+\cdots)=-(hD+h^2D^2/2+\cdots).$$

1.4 If $f$ is of degree less than $n$, then

$$(1-\sum_a e^{a_1D}+\sum_a e^{(a_1+a_2)D}-\sum_a e^{(a_1+a_2+a_3)D}+\cdots)f(x)$$
$$=[(1-e^{a_1D})(1-e^{a_2D})\cdots(1-e^{a_nD})]f(x)=0,$$

that is,

$$(1+\sum_a e^{(a_1+a_2)D}+\sum_a e^{(a_1+a_2+a_3+a_4)D}+\ldots)f(x)=(\sum_a e^{a_1D}+\sum_a e^{(a_1+a_2+a_3)D}+\ldots)f(x),$$

or 1.41 $$\Sigma f(x+s)=\Sigma f(x+t),$$

where $s$ is the sum of any even number $(0, 2, 4, \cdots)$, of the numbers $a_1, a_2, a_3 \cdots, a_n$ and $t$ the sum of any odd number of them.

1.5 Let $a_1 = a_2 = \cdots = a_n = a$. Then 1.41 becomes the well-known algebraic identity

$$1.51 \quad \binom{n}{0} f(x) + \binom{n}{2} f(x+2a) + \binom{n}{4} f(x+4a) + \cdots$$

$$= \binom{n}{1} f(x+a) + \binom{n}{3} f(x+3a) + \binom{n}{5} f(x+5a) + \cdots,$$

where $f$ is of degree less than $n$. If we let $f(x) = 1$, 1.51 becomes equivalent to $(1-1)^n = 0$.

(If $f$ is of degree $n$ with leading coefficient unity, the analogue of 1.41 would be

$$\Sigma f(x+s) = \Sigma f(x+t) + (-1)^n a_1 a_2 \cdots a_n D^n f(x),$$

or

$$\Sigma f(x+s) = \Sigma f(x+t) + (-1)^n a_1 a_2 \cdots a_n (n!).$$

In this case $(-a)^n(n!)$ must be added to the right member of 1.51. This supplement may be applied to each use of 1.51 in what follows.)

2. *The Tarry-Escott Problem.* The problem is to find two sets of numbers which have the same sum of $k$th powers for $k = 1, 2, 3, \cdots, m$. The notation generally used to express this property is

$$a_1, \cdots, a_r {}^m b_1, \cdots, b_s,$$

or $(a_1, \cdots, a_r; b_1, \cdots, b_s)$ is a set of degree $m$.*

If we let $f(x)$ equal in turn $x, x^2, x^3, \cdots, x^{n-1}$, all of degree less than $n$, then 1.41 becomes Escott's Theorem (Dorwart and Brown's paper, p. 617):

2.1 If we take as the $b$'s the number zero and all possible sums of an even number of the numbers $a_i$ and as the $c$'s all possible sums of an odd number of the numbers $a_i (i = 1, 2, 3, \cdots, n)$, then $(x+b_k, \cdots; x+c_\alpha, \cdots)$ is a set of degree $n-1$.

Dickson devotes a chapter of his *History of the Theory of Numbers*, II, p. 705, to the Tarry-Escott problem. Most but not all of the solutions he cites are special cases of 2.1. They are listed below in groups according to degree, the examples in each group in order of

*This paper touches at several points two papers which appeared in *American Mathematical Monthly* in 1937, Vol. 44: "The Tarry-Escott Problem", by H. L. Dorwart and O. E. Brown, pp. 613-626; and "Ideal Solutions of the Tarry-Escott Problem", by Jack Chernick, pp. 626-633.

their appearance in the book. After substitution in 2.1, it is often necessary to cancel like terms, one on each side, and sometimes to drop a zero on one side.

It will be noticed that many sets of $a$'s resemble in whole or in part the Fibonacci series: 1, 2, 3, 5, 8, 13, $\cdots$, in that two $a$'s of a group are often added to produce a third one; this produces many pairs of like terms, one on each side, which cancel in the solution.

If we let $x=0$, $a_i=d_i r^{i-1}(i=1, 2, 3, \cdots, n;\ d_i<r)$, then 2.1 may be stated in this form:

2.2 From the number in any number-scale whose digits are $d_1, d_2, \cdots, d_n$, form all possible numbers in which an even number $(0, 2, 4, \cdots)$ of the $d$'s is replaced by zeros and all those in which an odd number of the $d$'s is replaced by zeros; then the two groups of numbers form a set of degree $n-1$.

If each $d$ is unity, then (0, 11, 101, 110; 1, 10, 100, 111) is a set of degree 2; (0, 11, 101, 110, 1001, 1010, 1100, 1111; 1, 10, 100, 1000, 111, 1011, 1101, 1110) is a set of degree 3; etc. If these sets are regarded as written in the binary scale and are then changed to the decimal scale, the results are:

2.3 (0, 3, 5, 6; 1, 2, 4, 7) is a set of degree 2;

(0, 3, 5, 6, 9, 10, 12, 15; 1, 2, 4, 7, 8, 11, 13, 14) is a set of degree 3; etc.

The second of these is essentially A. Gerardin's solution: $x$, $x+3$, etc. (11th in our list) in Dickson's *History*, II, p. 708. These may be derived directly from 2.1 by letting $x=0$, $a_i=2^{i-1}$, and may be stated in this form:

2.4 Write in the binary scale the $2^n$ consecutive integers beginning with 0; separate them into two groups, one in which the sum of the digits of each number is even, one in which it is odd; then the two groups form a set of degree $n-1$.

The theorem 2.1 is known not to be *the* general solution of the Tarry-Escott problem. It gives only solutions in which the numbers may be separated into pairs each of which has a constant sum $(2x+\Sigma a_i)$, and not even all of these. But there are examples in Dickson's *History*, II, pp. 705 seq., in which the numbers may not be so paired off, e. g.:

$(f(a,b), f(c,d), f(a+c,b+d); f(a,c), f(b,d), f(a+b,c+d))$ of degree 2, where $f(x,y)=x^2+xy+y^2$ (F. Proth, p. 705);

$(ab+cd, bc, ad; bc+ad, ab, cd)$ of degree 2 (R. W. Christie, p. 706;

| Substitution in General Solution 2.1 | | | | | | | Beginning of Special Solution | Author Page | Degree |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | | | |
| $\frac{a+b+c}{3}$ | $\frac{2a-b-c}{3}$ | $\frac{-a+2b-c}{3}$ | $\frac{-a-b+2c}{3}$ | | | | $a,b,c =$ | L. Dickson, 705 | |
| $\delta$ | $\alpha$ | $\beta$ | $\gamma$ | | | | $\alpha+\beta+\delta,\alpha+\gamma+\delta,\ldots$ | C. Goldbach, 705 | |
| 0 | $a$ | $b$ | $c$ | | | | $a_1b_1c =$ | L. Euler, 705 | |
| $p+a+n$ | $-a-2n$ | $n$ | $a+n$ | | | | $p,p+a,\ldots$ | F. Pollock, 705 | 2 |
| 0 | $a$ | $b$ | $a+b$ | | | | $a,b,2a+2b =$ | A. Martin, 706 | |
| 0 | $p$ | $q$ | $2p+2q$ | | | | $p,q,2p+2q =$ | A. Martin, 706 | |
| 0 | $a+b-c$ | $a-b+c$ | $-a+b+c$ | | | | $a+b+c,\ a+b-c,\ldots$ | A. Martin, 706 | |
| 0 | $a$ | $b$ | $a+b$ | $a+2b$ | | | $3a+3b,2a+4b,\ldots$ | J. W. Nicholson, 706 | |
| 1 | 1 | 2 | 3 | 5 | | | $1,5,8,12 = 2,3,10,11$ | M. Frolov, 706 | |
| 1 | 1 | $g$ | $g+1$ | $2g+1$ | | | $1,g+3,\ldots$ | A. Gerardin, 707 | |
| $x$ | 1 | 2 | 4 | 8 | | | $x,x+3,\ldots$ | A. Gerardin, 708 | 3 |
| $a-b$ | $b+c$ | $-c$ | $b$ | $b-c$ | | | $a-b,a-2c,\ldots$ | C. Bisman, 709 | |
| 2 | 1 | 2 | 3 | 9 | | | $3,4,8,\ldots$ | E. N. Barisien, 713 | |
| 0 | $a$ | $b$ | $a+b$ | $a+2b$ | $a+3b$ | $a+4b$ | $5a+10b,4a+11b,\ldots$ | J. W. Nicholson, 706 | |
| $a-32$ | 2 | 4 | 6 | 10 | 16 | 26 | $a=32,a=24,\ldots$ | J. W. Nicholson, 706 | |
| 2 | 1 | 6 | 7 | 8 | 9 | 15 | $2,12,15,\ldots$ | A. Gerardin, 707 | 5 |
| $c$ | $a-b-c$ | $b$ | $a-c$ | $a+b-c$ | $a+2b-c$ | $2a+b-2c$ | $c,a+3b,\ldots$ | G. Tarry, 708 | |
| $n$ | 1 | 2 | 3 | 4 | 5 | 7 | $1+n,2+n,\ldots$ | E. Miot, 710 | |
| $5a-4b-5c$ | $-a+2c$ | $a+b-3c$ | $b-c$ | $-a+b+c$ | $-2a+b+3c$ | $-a+2b$ | $6a-3b-8c,5a-9c,\ldots$ | G. Tarry, 710 | |

(2, 4, 20, 22, 23; 1, 6, 16, 26, 32) of degree 3, and others (A. Gerardin, p. 707);

(14, 23, 25, 138; 7, 26, 30, 137) of degree 2 (A. Gerardin, p. 707);

$(2d+3x,\ 4d+2x,\ d;\ d+2x,\ 4d+3x,\ 2d)$ of degree 2 (A. Gerardin, p. 708).

Perhaps there is a multiplicative theorem by means of which solu tion of the same degree generate others of greater complexity. There foll ow a solution in which the numbers may be paired off yet which is not given by 2.1. The identity

$$\tfrac{1}{2}[p^2+q^2+(p+q)^2] \equiv p^2+pq+q^2 = r^2+rs+s^2 \equiv \tfrac{1}{2}[r^2+s^2+(r+s)^2],$$

implying

$$\tfrac{1}{2}[p^4+q^4+(p+q)^4] \equiv (p^2+pq+q^2)^2 = (r^2+rs+s^2)^2 \equiv \tfrac{1}{2}[r^4+s^4+(r+s)^4],$$

permits us to assume a solution of the 5th degree of the form:

$$(p,\ q,\ p+q,\ -p,\ -q,\ -p-q;\ r,\ s,\ r+s,\ -r,\ -s,\ -r-s)$$

if $$p^2+pq+q^2 = r^2+rs+s^2.$$

One solution of this last equation is $p=a-b,\ q=2a+5b,\ r=a+4b,\ s=2a+b$, giving $\pm(a-b,\ 2a+5b,\ 3a+4b;\ a+4b,\ 2a+b,\ 3a+5b)$ of degree 5 (wherein each number is taken both positively and negatively), which however is simply another special case of 2.1 with $x=-3a-5b$, $a_1=a,\ a_2=b,\ a_3=a+b,\ a_4=a+2b,\ a_5=a+3b,\ a_6=2a+3b$. Another solution of $p^2+pq+q^2=r^2+rs+s^2$ which does *not* result in a special case of 2.1 is $p=a^2,\ q=b^2,\ r=a^2-ab-b^2,\ s=2ab$, giving $\pm(a^2,\ b^2,\ a^2+b^2;\ a^2-ab-b^2,\ 2ab,\ a^2+ab-b^2)$ of degree 5, simpler than the 2-parameter solutions of this degree given in Chernick's above-mentioned paper, *American Mathematical Monthly*, 1937, p. 630. This would give for instance $\pm(4,\ 25,\ 29;\ 11,\ 20,\ 31)$ of degree 5, which by addition of 32 to each term becomes (1, 12, 21, 43, 52, 63; 3, 7, 28, 36, 57, 61) of degree 5 (E. Aubrey by a different method, Dickson, II, p. 712.)

3. *Further References.* Special cases of 1.51 and 2.1 differing only in form, have appeared in the literature with proofs more difficult than the one here given for the general theorem. Identity 1.51 is proved in Boole's *Treatise on the Calculus of Finite Differences*, p. 36, by a longer process employing another operator ($\Delta$). It may also be proved with greater difficulty from the relationship

$$\sum_{k=0}^{n} f(x+ka)/g'(x+ka) = 0,$$

where $f(x)$ is of degree less than $n$ and $g(x)$ has the $n+1$ roots: $x$, $x+a$, $x+2a, \cdots, x+na$. The special case of 1.51 in which $f(x)=x^p$ may be proved by equating coefficients of $y^p/(p!)$ on both sides of the identity:

$$\binom{n}{0}e^{xy}-\binom{n}{1}e^{(x+a)y}+\binom{n}{2}e^{(x+2a)y}\cdots=e^{xy}(1-e^{ay})^n$$

$$(p=1, 2, 3, \cdots, n-1).$$

From this case by the addition of functions of the type $cx^p$, we may return again to the general equation 1.51. Special cases of 1.51, with $a=1$, $n>1$, $x=0$, $f(x)=x$ of degree $1<n$, and with $a=1$, $n=1$, $f(x)=1$ of degree $0<n$, are proved (in different form) in Polya and Szegö's *Aufgaben und Lehrsätze*, I, pp. 6 and 159, by means of differentiation of an algebraic identity.

Systems of the type of 2.3 are proved by Graham in the *American Mathematical Monthly*, 1939, p. 173, by equating coefficients of $x^p/(p!)$ in the identity:

$$1-(1-e^x)(1-e^{2x})\cdots(1-e^{2^n-x})=e^x+e^{2x}-e^{3x}+\cdots\pm e^{(2^n-1)x}.$$

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# On the Motion of a Rigid Body

By VINCENT C. POOR
*University of Michigan*

1. *Introduction.* In the development of the equations of motion of a rigid body coordinates are usually introduced some place in the argument. The reason vector methods are not used exclusively is due to the seeming difficulty in defining instantaneous angular velocity. It is the purpose of this note to develop the equation of motion of a rigid body by absolute geometric methods. The ultimate objective is to arrive at the equation of relative motion, which has place in meteorological problems. For completeness the notions of instantaneous axis and angular velocity will be reassembled here.

2. *Instantaneous Angular Velocity.* Consider a rigid body with a fixed point. It will be shown first that the body has an instantaneous axis of rotation, that is, there exists a straight line every point of which has a zero instantaneous linear velocity.

Let $A$, a point of a rigid body, be fixed in space. Let $V_p$ be the instantaneous linear velocity of a point $P$ fixed in the body. Then $V_p$ is perpendicular to $AP$ otherwise there would be a component velocity tending to separate $P$ from $A$, which is impossible since the body is rigid. Pass a plane through $AP$ perpendicular to $V_p$. The velocity of every point, $P_1$, of this plane is parallel to $V_p$ otherwise there would be a component velocity separating $P_1$ from $P$. Hence the instantaneous velocity of every point of this plane is perpendicular to the plane. Select, $Q$ a point fixed in the body and not in this plane; pass a plane through $AQ$ perpendicular to $V_Q$, the instantaneous linear velocity of $Q$. Then by the previous argument the velocity of every point of this plane is perpendicular to the plane. These two planes if $Q$ is properly chosen, intersect in a straight line. The velocity of a point $R$ in this line is thus perpendicular to three different directions it is therefore zero and the line $AR$ is a line instantaneously at rest. This is the instantaneous axis of rotation.

We may now define instantaneous angular velocity; if $R$ is the foot of the perpendicular from $P$ to $AR$ then $|V_p|$ divided by the length of $AR$ is defined to be the magnitude of the instantaneous angular velocity. We call the angular velocity itself $\omega$ and lay it off as a vector on the axis of rotation. Then from elementary mechanics

$$V_P = \omega \times \overrightarrow{AP} \tag{0}$$

is the vector equation defining instantaneous angular velocity.

3. *The Rigid Body with a Fixed Point.* We are thinking of a frame of reference fixed in space in the classical sense possibly a tetrahedron, and a frame fixed in the body. Let $O$ be a point fixed in space and $A$ a point in the body. Let $P$ be any other point of the body. Then we write the vector equation

$$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP} \tag{1}$$

and using Newton's fluxion notation for time derivatives we have

$$\dot{\overrightarrow{OP}}=\dot{\overrightarrow{OA}}+\dot{\overrightarrow{AP}}. \tag{2}$$

But $\dot{\overrightarrow{OA}}$ is the translational velocity of the body. Thus if $\dot{\overrightarrow{OA}}$ be transposed to the left member of the equation $\dot{\overrightarrow{AP}}$ will be the velocity of a point $P$ of a rigid body with the fixed point $A$. That is if a body in motion has no translational velocity it has at least one fixed point. A substitution of

$$\omega\times\overrightarrow{AP} \quad \text{for} \quad \dot{\overrightarrow{AP}} \quad \text{makes}$$

$$\dot{\overrightarrow{OP}}=\dot{\overrightarrow{OA}}+\omega\times\overrightarrow{AP}. \tag{3}$$

In the sequel we will frequently think of the term $\dot{\overrightarrow{OA}}$ as having been transposed to the left member of (2) without stating it. It is in this sense that we are studying or applying the motions of a rigid body with a fixed point.

Let $W$ be a vector drawn from the point $A$ of the rigid body referred to a frame in fixed space and let $W_r$ be its "*alter-ego*" the same vector referred to a frame fixed in the body. Then if the tip of $W$ is a fixed point of the body, by (0) then,

$$\dot{W}=\omega\times W,$$

if there is only a rotational motion of the body. If now the tip of $W$ also moves through the body, to $\omega\times W_r$ must be added $\dot{W}_r$ the velocity of the tip of $W$ relative to the frame fixed in the body. We thus have

$$\dot{W}=\dot{W}_r+\omega\times W_r \tag{4}$$

4. *The Dynamics of a Rigid Body.* In equation (3) $P$ is a point fixed in the body. If $P$ moves through the body its absolute velocity $V$ must be added to the right member of (3); thus

$$\dot{\overrightarrow{OP}}=\dot{\overrightarrow{OA}}+V+\omega\times\overrightarrow{AP} \tag{5}$$

The time derivative of (5) furnishes

$$\ddot{\overrightarrow{OP}} = \ddot{\overrightarrow{OA}} + \dot{V} + \dot{\omega} \times \overrightarrow{AP} + \omega \times \dot{\overrightarrow{AP}} \tag{6}$$

But $\dot{V} = \dot{V}_r + \omega \times V_r$ by (4). By a further application of (4) $\dot{\overrightarrow{AP}} = V_r + \omega \times \overrightarrow{AP}$. A substitution of these results into (6) gives

$$\ddot{\overrightarrow{OP}} = \ddot{\overrightarrow{OA}} + \dot{V}_r + \omega \times V_r + \dot{\omega} \times \overrightarrow{AP} + \omega \times (V_r + \omega \times \overrightarrow{AP}) \tag{7}$$

which reduces to

$$\ddot{\overrightarrow{OP}} = \ddot{\overrightarrow{OA}} + \dot{V}_r + \dot{\omega} \times \overrightarrow{AP} + \omega \cdot \overrightarrow{AP}\omega - \omega^2 \overrightarrow{AP} + 2\omega \times V_r \tag{8}$$

where $\omega \times (\omega \times \overrightarrow{AP})$ has been replaced by its expanded form. When this last equation is multiplied by the mass $m$ situated at the point $P$ this will be the dynamical equation of motion of $m$.

5. *Relative Motion.* If there is no external force acting on $m$ at $P$, $\ddot{\overrightarrow{OP}} = 0$ and if there is no translational motion or if $\dot{\overrightarrow{OA}}$ is constant

$$\ddot{\overrightarrow{OA}} = 0; \quad (8) \text{ then reduces to}$$

$$\dot{V}_r = \omega^2 \overrightarrow{AP} - \dot{\omega} \times \overrightarrow{AP} - \omega \cdot \overrightarrow{AP}\omega - 2\omega \times V_r \tag{9}$$

This is the equation of relative motion. The first term on the right multiplied by $m$ is the centrifugal force, $-2\omega \times V_r$ is the acceleration of Coriolis, while the other terms characterize the rigid body motion.

For relative motion on the earth $\omega$ is constant while if $A$ is taken as the foot of the perpeneicular from $P$ to the axis, then

$$\omega \cdot \overrightarrow{AP} = 0,$$

thus the first two negative terms vanish. There is an external force, the gravitational attraction which is not zero. If $O$ the point fixed in space is taken at the center of the earth then

$$m\ddot{\overrightarrow{OP}} = -mg\frac{\overrightarrow{OP}}{R}$$

where $R$ is $|\overrightarrow{OP}|$. When $\ddot{\overrightarrow{OP}}$ is replaced by its value from the last equation (8) becomes

$$\dot{V}_r = -g\frac{\overrightarrow{OP}}{R} + \omega^2 \overrightarrow{AP} - 2\omega \times V_r.$$

For the drift of a bullet, say, shot due north in the Northern hemisphere, neither the gravitational attraction nor the centrifugal force plays any role. The drift is thus characterized by the acceleration of Coriolis. This gives some idea of the importance of the acceleration of Coriolis in meteorological problems.

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# TO OUR SUBSCRIBERS

Due to the fact that the present Editor-and-Manager has for more than a year been without any office help and has, besides, for many months been busy teaching several hours a day, it has been impossible for him to send notices to the subscribers when their subscriptions have expired. He has, however, from time to time published notices in the MAGAZINE requesting subscribers not to wait for formal statements before renewing their subscriptions. In response to these notices practically no requests for cancellation of subscriptions have been received.

In justice to the Editor-and-Manager it should not be forgotten by any one of the hundreds of friends of the MAGAZINE that when in 1942 Louisiana State University discontinued its sponsorship he undertook with the magnificent help of his editorial colleagues to carry on, even placing his name on the title page as publisher. Due to the desperate illness of E. J. Land of the Franklin Press, in Baton Rouge, La., an illness covering a period of four months or more, there was no one in that splendid printing firm to handle the Monotype machine, which, for the past nineteen years has been used to set the MAGAZINE. It was because of this situation that the December issue was delayed for nearly three months, and the January issue was not off the press until April.

Accomplishing one of the most remarkable come-backs the writer has ever witnessed, Mr. Land, though working part-time, has to date printed all issues due you in Volume XIX and with a typographical finish which we believe is not excelled.

If you have been getting the MAGAZINE over a period not as yet covered by your renewal check, and, if you wish to *continue* your subscription to it, remittance from you to cover a year, or even more than a year, in addition to past months will be welcomed as both a practical help to the MAGAZINE, and, what is equally valuable, an inspiration to the Editor-and-Manager. Many subscribers have paid several years in advance.

Our printer's debt will be a large one this year. Every paid-in dollar that is due the MAGAZINE will be definitely needed in discharging the debt.

S. T. SANDERS.
*Editor-and-Manager.*

# A Systematic Method of Finding Pythagorean Numbers

By MERTON TAYLOR GOODRICH
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We start with the well known relationships $a=p^2-q^2$, $b=2pq$, $c=p^2+q^2$. If $p$ and $q$ are both odd, they differ by a multiple of 2, and $a$, $b$, and $c$ contain a common factor 2. Also, if $p$ and $q$ are not relatively prime, $a$, $b$, and $c$ contain a common factor. We therefore restrict $p$ and $q$ to be relatively prime positive integers, not both odd, $p$ being greater than $q$. We define a primitive set of pythagorean numbers as one which contains no common factor. Then $a$, $b$, and $c$, as above defined, is a primitive set of positive pythagorean numbers, and conversely, if $a$, $b$, and $c$ is a primitive set of positive pythagorean numbers, there exist positive integers $p$ and $q$, relatively prime, not both odd, $p$ greater than $q$, such that these formulas hold.

Since $p$ is greater than $q$, $p=q+d$, $d$ being a positive integer. Since $p$ and $q$ are relatively prime, and not both odd, $d$ is odd and relatively prime to both $p$ and $q$.

Substituting this value for $p$ in these formulas, we have:

$$a=q^2+2qd+d^2-q^2=d(2q+d)$$

$$b=2q(q+d)$$

$$c=q^2+2qd+d^2+q^2=2q^2+2qd+d^2=b+d^2.$$

These formulas may be used to find the values of $a$, $b$, and $c$, but there is an easier way. We may assemble the sets in groups so that each group contains sets with a common value of $d$. In each group, we arrange the sets so that the values of $q$ form a series of consecutive numbers, but we compute sets only for those values of $q$ relatively prime to $d$. Then we may place the numbers in a table with columns $d$, $q$, $a$, $b$, and $c$, and from this table obtain the following relationships.

If $q$ is 1 in any group, $a=d(d+2)$. This means that in the first set of each group, $a$ is the product of two consecutive odd numbers, the first of which is $d$.

Also, the terms in the "a" column form a series $d^2+2d$, $d^2+4d$, $d^2+6d$, ····· with a common difference of $2d$.

Similarly, the terms in the "b" column form a series $2d+2$, $4d+8$, $6d+18$, $8d+32$, ····· with a series of first differences $2d+6$,

$2d+10$, $2d+14$, ····· with a common difference of 4, independent of the value of $d$.

For the "c" column, the numbers are all equal to $b+d^2$, and hence the differences are the same as the corresponding ones in the "b" column.

We now insert in the table three more columns, the first consisting of the values of $2d$, the second consisting of the constant second difference for the values of $b$ and $c$, namely 4, which we call $e$ and the third consisting of the series of first differences for $b$ and $c$, which we call $f$.

The first group shows that there is at least one primitive set of pythagorean numbers for every odd value of $a$ above one. Since the values of $b$ start with 4 when $d=1$, and the starting value of $b$ in each group increases by 4 from group to group, there is at least one set of pythagorean numbers for every number divisible by 4. But for alternate even numbers beginning with 2, such as 2, 6, 10, 14, ·····, there is no primitive set. There is no set, primitive or imprimitive, which contains the numbers 1 or 2.

Therefore, all the primitive sets of positive pythagorean numbers may be found by constructing a table according to this plan. By this method there is no chance of skipping a set or duplicating a set. The processes used include only the multiplication and addition of comparatively small numbers.

In each group, $d=1, 3, 5, 7, \cdots\cdots$. If subscripts denote the order of the set in each group, we have the formulas:

$$a_1=d(d+2) \qquad b_1=2(d+1) \qquad c_1=b_1+d^2=a_1+2$$

$$a_2=a_1+2d \qquad b_2=b_1+f_2 \qquad c_2=c_1+f_2=b_2+d^2$$

····· ····· ·····

The following is a sample table in which the values of $d$ and $q$ are each less than 10.

*TABLE OF PRIMITIVE SETS OF PYTHAGOREAN NUMBERS*

| $d$ | $q$ | $2d$ | $e$ | $f$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 4 | 3 | 4 | 5 |
| 1 | 2 | 2 | 4 | 8 | 5 | 12 | 13 |
| 1 | 3 | 2 | 4 | 12 | 7 | 24 | 25 |
| 1 | 4 | 2 | 4 | 16 | 9 | 40 | 41 |
| 1 | 5 | 2 | 4 | 20 | 11 | 60 | 61 |
| 1 | 6 | 2 | 4 | 24 | 13 | 84 | 85 |
| 1 | 7 | 2 | 4 | 28 | 15 | 112 | 113 |
| 1 | 8 | 2 | 4 | 32 | 17 | 144 | 145 |
| 1 | 9 | 2 | 4 | 36 | 19 | 180 | 181 |

| $d$ | $q$ | $2d$ | $e$ | $f$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 6 | 4 | 8 | 15 | 8 | 17 |
| 3 | 2 | 6 | 4 | 12 | 21 | 20 | 29 |
| 3 | 3 | 6 | 4 | 16 | * | * | * |
| 3 | 4 | 6 | 4 | 20 | 33 | 56 | 65 |
| 3 | 5 | 6 | 4 | 24 | 39 | 80 | 89 |
| 3 | 6 | 6 | 4 | 28 | * | * | * |
| 3 | 7 | 6 | 4 | 32 | 51 | 140 | 149 |
| 3 | 8 | 6 | 4 | 36 | 57 | 176 | 185 |
| 3 | 9 | 6 | 4 | 40 | * | * | * |
| 5 | 1 | 10 | 4 | 12 | 35 | 12 | 37 |
| 5 | 2 | 10 | 4 | 16 | 45 | 28 | 53 |
| 5 | 3 | 10 | 4 | 20 | 55 | 48 | 73 |
| 5 | 4 | 10 | 4 | 24 | 65 | 72 | 97 |
| 5 | 5 | 10 | 4 | 28 | * | * | * |
| 5 | 6 | 10 | 4 | 32 | 85 | 132 | 157 |
| 5 | 7 | 10 | 4 | 36 | 95 | 168 | 193 |
| 5 | 8 | 10 | 4 | 40 | 105 | 208 | 233 |
| 5 | 9 | 10 | 4 | 44 | 115 | 252 | 277 |
| 7 | 1 | 14 | 4 | 16 | 63 | 16 | 65 |
| 7 | 2 | 14 | 4 | 20 | 77 | 36 | 85 |
| 7 | 3 | 14 | 4 | 24 | 91 | 60 | 109 |
| 7 | 4 | 14 | 4 | 28 | 105 | 88 | 137 |
| 7 | 5 | 14 | 4 | 32 | 119 | 120 | 169 |
| 7 | 6 | 14 | 4 | 36 | 133 | 156 | 205 |
| 7 | 7 | 14 | 4 | 40 | * | * | * |
| 7 | 8 | 14 | 4 | 44 | 161 | 240 | 289 |
| 7 | 9 | 14 | 4 | 48 | 175 | 288 | 337 |
| 9 | 1 | 18 | 4 | 20 | 99 | 20 | 101 |
| 9 | 2 | 18 | 4 | 24 | 117 | 44 | 125 |
| 9 | 3 | 18 | 4 | 28 | * | * | * |
| 9 | 4 | 18 | 4 | 32 | 153 | 104 | 185 |
| 9 | 5 | 18 | 4 | 36 | 171 | 140 | 221 |
| 9 | 6 | 18 | 4 | 40 | * | * | * |
| 9 | 7 | 18 | 4 | 44 | 207 | 224 | 305 |
| 9 | 8 | 18 | 4 | 48 | 225 | 272 | 353 |
| 9 | 9 | 18 | 4 | 52 | * | * | * |

*$d$ and $q$ not being relatively prime, this is not a primitive set, and is not computed.

This table may be continued indefinitely, and there may be found as many primitive sets of pythagorean numbers as desired. By multiplying all the terms in each set by all the integers all the pythagorean numbers may be found.

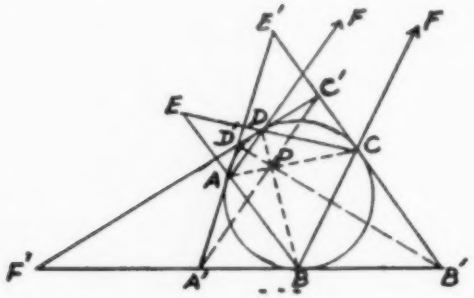
# Some New Properties of the Triangle

By C. S. Lu (Lu Chin-Shih)
*Chungking, China*

In any triangle the midpoints of the sides, the feet of the altitudes, the midpoints of the lines drawn from the vertices to the orthocenter, these nine points lie on a circle which is called the nine-point circle of the triangle. The center of this circle lies midway between the circumcenter and the orthocenter and, therefore, on the Euler line of the triangle. If tangents are drawn to the nine-point circle at these nine points, we may discover some marvellous properties of the triangle which bear close relation to the Euler line.

In order to state these properties let us introduce in advance the following two lemmas, the latter one being the combination of Pascal's Theorem and Brianchon's Theorem.

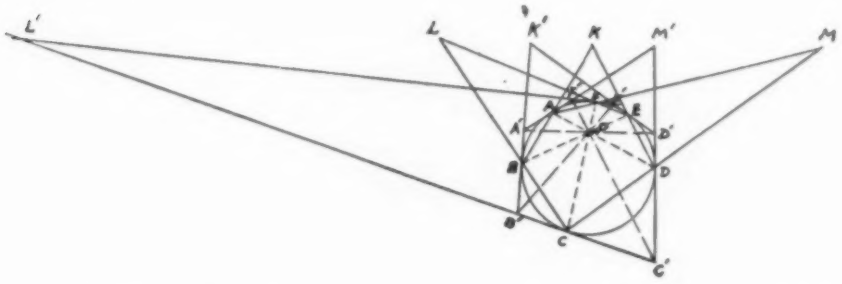
Lemma 1. *If a circumscribed quadrilateral is formed by the tangents to a circle at the vertices of an inscribed quadrilateral, the diagonals of these quadrilaterals are concurrent at a pole and the points of intersection of the opposite sides of these quadrilaterals are collinear on the polar.*




In the complete quadrilateral $ABCDEF$ the diagonal $EF$ is the polar of the point of intersection $P$ of the diagonals $AC$ and $BD$. Now $AA'$, $BA'$ are the tangents at $A$, $B$ such that $AB$ is the polar of $A'$, similarly $CD$ is the polar of $C'$. Hence $A'C'$ is the polar of the point of intersection $E$ of $AB$, $CD$. Now the polar $EF$ of $P$ passes through $E$; it follows that the polar $A'C'$ of $E$ passes through $P$. In like manner it may be shown that the polar $B'D'$ of $F$ passes through $P$. Consequently, the diagonals $AC$,$BD$,$A'C'$,$B'D'$ are concurrent at the pole $P$.

Moreover, the points $E',F',E,F$ are the poles of $AC,BD,A'C',B'D'$. Hence they are collinear since their polars are concurrent.

Lemma II. *If the diagonals of an inscribed hexagon are concurrent at a pole, the points of intersection of the opposite sides are collinear on the polar, and in the circumscribed hexagon formed by the tangents at the vertices of the inscribed hexagon the diagonals are concurrent at the same pole and the points of intersection of the opposite sides are collinear on the same polar.*

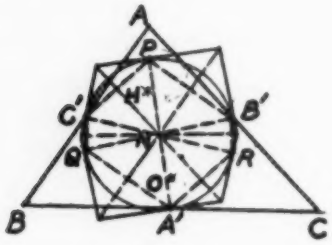


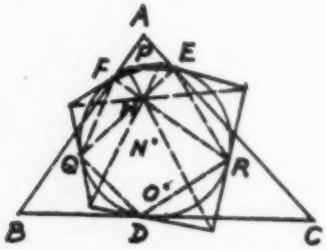

$AD,BE$ are the diagonals of the inscribed quadrilateral $ABDE$, and $A'D'$ is one of the diagonals of the circumscribed quadrilateral formed by the tangents at $A,B,D,E$. Hence, by Lemma I, the diagonals $AD$, $BE$, $A'D'$ are concurrent at a point $P$ and the points of intersection $K$, $K'$, $M'$ of the opposite sides of these quadrilaterals are collinear on the polar of $P$. In like manner it may be shown that all the diagonals $AD,BE,CF,A'D',B'E',C'F'$ are concurrent at the same point $P$ and the points of intersection $K,L,M,K',L',M'$ are collinear on the same polar of $P$.

Now we begin to state the marvellous properties of the triangle, which are arranged in the form of a sequence of theorems and corollaries. For the convenience of the reader, we use the following notation:

| | |
|---|---|
| $A,B,C,$ | denote the vertices of the triangle; |
| $A',B',C',$ | the midpoints of the sides $BC,CA,AB;$ |
| $D,E,F,$ | the feet of the altitudes $AD,BE,CF;$ |
| H, | the orthocenter; |
| $P,Q,R,$ | the midpoints of the lines $AH,BH,CH;$ |
| $O,$ | the circumcenter; |
| $N,$ | the center of the nine-point circle. |

Theorem I. *In the circumscribed hexagon formed by the tangents to the nine-point circle $(N)$ at $P,Q,R,A',B',C'$, the three diagonals are concurrent at the center $N$ of the nine-point circle.*



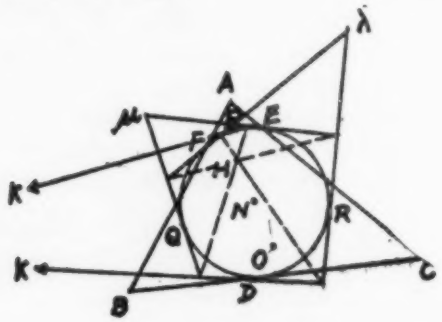




In the inscribed hexagon $PQRA'B'C'$ the diagonals are concurrent at $N$ since they are diameters of $(N)$. Hence, by Lemma II, the diagonals of the circumscribed hexagon are concurrent at the same point $N$.

Theorem II. *In the circumscribed hexagon formed by the tangents to the nine-point circle $(N)$ at $P,Q,R,D,E,F$, the three diagonals are concurrent at the orthocenter $H$.*

In the inscribed hexagon $PQRDEF$ the three diagonals are concurrent at $H$ since they are altitudes of the triangle $ABC$. Hence, by Lemma II, the diagonals of the circumscribed hexagon are concurrent at the same point $H$.

Theorem III. *In the circumscribed hexagon formed by the tangents to the nine-point circle $(N)$ at $P,Q,R,D,E,F$, the points of intersection $\kappa,\lambda,\mu$ of the opposite sides are collinear, the common line being perpendicular to the Euler line of the triangle.*



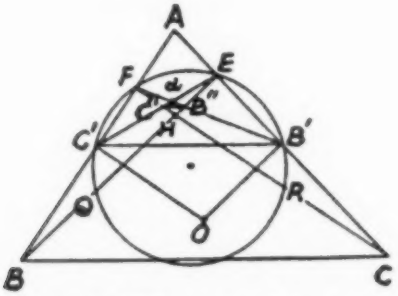
By Theorem II, the three diagonals of the circumscribed hexagon are concurrent at the orthocenter $H$. Hence, by Lemma II, the points of intersection $\kappa,\lambda,\mu$ of the opposite sides of the hexagon are collinear on the polar of $H$. Now the polar of $H$ is perpendicular to the Euler line of the triangle. Hence the proposition.

Theorem IV. *In the inscribed hexagon PQRDEF the points of intersection $\kappa',\lambda',\mu'$ of the opposite sides are collinear, the common line being perpendicular to the Euler line of the triangle.*

It may be shown in a similar way.

Corollary. *In the inscribed hexagon PQRDEF and the circumscribed hexagon formed by the tangents at P,Q,R,D,E,F, the points intersection $\kappa,\lambda,\mu,\kappa',\lambda',\mu'$ of the opposite sides of these hexagons are collinear on the polar H. and, therefore, they lie on the same line perpendicular to the Euler line of the triangle.*

Theorem V. *In the inscribed quadrilateral EFB'C', the diagonals EC' and FB' intersect at a point $\alpha$ which lies on the Euler line of the triangle.*

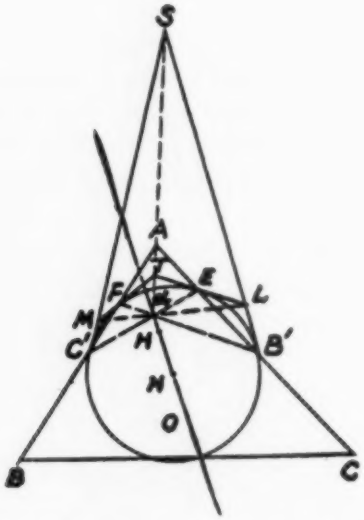



The angles $C'EQ$ and $B'FR$ are equal since they subtend equal arcs $C'Q$ and $B'R$ on the nine-point circle $(N)$. In similar triangles $\alpha FC''$ and $\alpha EB''$, $\alpha F : \alpha E = \alpha C'' : \alpha B''$; in similar triangles $\alpha FC'$ and $\alpha EB'$, $\alpha F : \alpha E = \alpha C' : \alpha B'$. Hence $\alpha C'' : \alpha B'' = \alpha C' : \alpha B'$, and $C''B''$ is parallel to $C'B'$. Now the similar triangles $\alpha C''B''$ and $\alpha C'B'$ are homothetic; so are the triangles $HC''B''$ and $OC'B'$, since their corresponding sides are parallel. Consequently, the quadrilateral $\alpha C''HB''$ and $\alpha C'OB'$ are homothetic figures, the homothetic center being the common vertex $\alpha$, Hence the corresponding vertices $H$ and $O$ are collinear with the homthetic center $\alpha$; i. e., the point of intersection $\alpha$ of $EC'$ and $FB'$ lies on the Euler line of the triangle.

Corollary I. *In the inscribed quadrilateral FDC'A', the diagonals FA' and DC' intersect at a point $\beta$ which lies on the Euler line of the triangle.*

Corollary II. *In the inscribed quadrilateral DEA'B', the diagonals DB' and EA' intersect at a point $\gamma$ which lies on the Euler line of the triangle.*

Theorem VI. *In the circumscribed quadrilateral LMST formed by the tangents to the nine-points circle (N) at E,F,B',C', the diagonals LM and ST intersect at the same point $\alpha$, which lies on the Euler line of the triangle.*



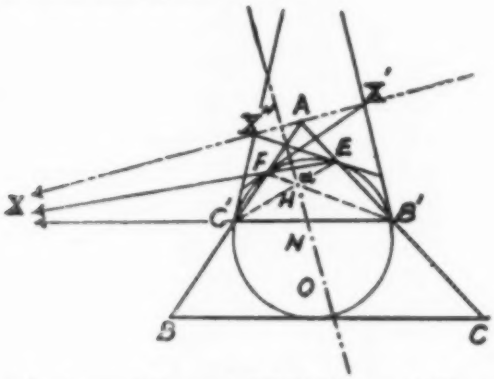

By Theorem V, the diagonals of the inscribed quadrilateral *EFB'C'* ntersect at a point $\alpha$ which lies on the Euler line. Hence, by Lemma I, the diagonals of the circumscribed quadrilateral intersect at the same point $\alpha$.

Corollary I. *In the circumscribed quadrilateral MKUV formed by the tangents to the nine-point circle (N) at F,D,C',A', the diagonals MK and UV intersect at the same point $\beta$ which lies on the Euler line of the triangle.*

Corollary II. *In the circumscribed quadrilateral KLJW formed by the tangents to the nine-point circle (N) at D,E,A',B', the diagonals KL and JW intersect at the same point $\gamma$ which lies on the Euler line of the triangle.*

Theorem VII. *In the circumscribed quadrilateral formed by the tangents to the nine-point circle (N) at E,F,B',C', the points of intersection X',X'' of the opposite sides lie on the line drawn from the vertex A of the triangle ABC to the point of intersection X of EF and B'C', and the*

*common line $AXX'X''$ is the polar of the point $\alpha$ and therefore it is perpendicular to the Euler line of the triangle.*




In the inscribed quadrilateral $EFB'C'$ $A$ and $X$ are the points of intersection of the opposite sides. By hypothesis, $X'$ and $X''$ are the points of intersection of the opposite sides of the circumscribed quadrilateral formed by the tangents at $E,F,B',C'$. Hence, by Lemma I, the points $A,X,X',X''$ are collinear on the polar of the point of intersection $\alpha$ of the diagonals of these quadrilaterals. Now, by Theorems V and VII, the point $\alpha$ lies on the Euler line. Hence its polar $AXX'X''$ is perpendicular to it.

Corollary I. *In the circumscribed quadrilateral formed by the tangents to the nine-point circle $(N)$ at $F,D,C',A'$, the points of intersection $Y',Y''$ of the opposite sides lie on the line drawn from the vertex $B$ of the triangle $ABC$ to the point of intersection $Y$ of $FD$ and $C'A'$, and the common line $BYY'Y''$ is the polar of the point $\beta$ and therefore is perpendicular to the Euler line of the triangle.*

Corollary I. *In the circumscribed quadrilateral formed by the tangents to the nine-point circle $(N)$ at $D,E,A',B'$, the points of intersection $Z',Z''$ of the opposite sides lie on the line drawn from the vertex $C$ of the triangle $ABC$ to the point of intersection $Z$ of $DE$ and $A'B'$, and the common line $CZZ'Z''$ is the polar of the point $\gamma$ and therefore is perpendicular to the Euler line of the triangle.*

Corollary III. *The lines $AXX'X''$, $BYY'Y''$, $CZZ'Z''$ are parallel to the line $\kappa\lambda\mu\kappa'\lambda'\mu'$.*

We have been occupied up to this point with the theorems concerning five sets of collinear points $\alpha,\beta,\gamma$; $\kappa,\lambda,\mu,\kappa',\lambda',\mu'$; $A,X,X',X''$; $B,Y,Y',Y''$; $C,Z,Z',Z''$, the former one lying on the Euler line and the latter four lying on lines perpendicular to the same Euler line of the triangle. There are, however, some more points which have close

connections with these points. They are considered in the following theorems and corollaries.

Theorem VIII. *The lines $FA'$,$DR$,$KL$ are concurrent at a point $\delta$.*
Corollary I. *The lines $DB'$,$EP$,$LM$ are concurrent at a point $\epsilon$.*
Corollary II. *The lines $EC'$,$FQ$,$MK$ are concurrent at a point $\zeta$.*

Theorem IX. *The lines $EA'$,$DQ$,$MK$ are concurrent at a point $\rho$.*
Corollary I. *The lines $FB'$,$ER$,$KL$ are concurrent at a point $\varsigma$.*
Corollary II. *The lines $DC'$,$FP$,$LM$ are concurrent at a point $\tau$.*

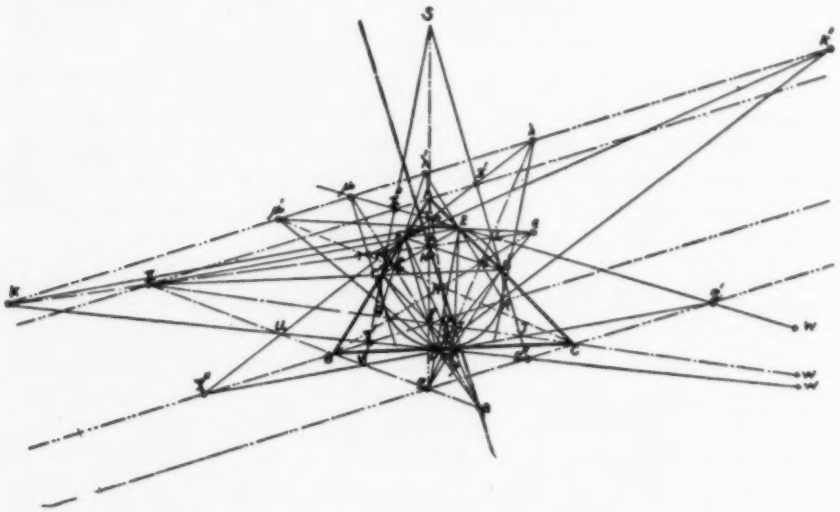
Theorem X. *The points $A$,$Y$,$Z$,$S$,$T$,$\alpha$ are collinear.*
Corollary I. *The points $B$,$Z$,$X$,$U$,$V$,$\beta$ are collinear.*
Corollary II. *The points $C$,$X$,$Y$,$J$,$W$,$\gamma$ are collinear.*

Theorem XI. *The points $A$,$N$,$\delta$,$\rho$,$\lambda'$ are collinear.*
Corollary I. *The points $B$,$N$,$\epsilon$,$\varsigma$,$\kappa'$ are collinear.*
Corollary II. *The points $C$,$N$,$\zeta$,$\tau$,$\mu'$ are collinear.*

Theorem XII. *The points $X$,$\alpha$,$\epsilon$,$\tau$,$\kappa$ are collinear.*
Corollary I. *The points $Y$, $\beta$,$\zeta$,$\rho$,$\mu$ are collinear.*
Corollary II. *The points $Z$,$\gamma$,$\delta$,$\varsigma$,$\lambda$ are collinear.*

Theorem XIII. *The line $AYZST\alpha$ is the polar of $X$.*
Corollary I. *The line $BZXUV\beta$ is the polar of $Y$.*
Corollary II. *The line $CXYJW\gamma$ is the polar of $Z$.*

Theorem XIV. *The line $X\alpha\epsilon\tau\kappa$ is the polar of $A$.*
Corollary I. *The line $Y\beta\zeta\rho\mu$ is the polar of $B$.*
Corollary II. *The line $Z\gamma\delta\varsigma\lambda$ is the polar of $C$.*



We have dealt with twenty-four points all of which are obtained by drawing tangents to the nine-point circle at these nine points. Besides those, there are other points which may be obtained from the same source. For instance, we may get another point $\Theta$ which is the common point of the concurrent lines $AK$,$BL$,$CM$. The study of these new points leads us to recall A. L. Crelle's words: "It is indeed wonderful that so simple a figure as the triangle is so inexhaustible in properties".


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# *The Teachers' Department*

*Edited by*
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## Perversion of Purpose in Elementary Mathematical Education

By WILLIAM L. SCHAAF
*Brooklyn College*

The history of mathematical education in this country during the past twenty-five years reveals the fact that, regardless of the terminology used, the admittedly possible outcomes of instruction in mathematics invariably fall into one or another of three basic types of changed behavior on the part of the learner, viz., *cultural*, *disciplinary*, and *utilitarian*. In short, learning mathematics potentially admits of (1) modification of attitudes, perspectives, appreciations and appraisals; (2) improvement in methods and effectiveness of thinking; and (3) achievement of knowledge and skills which may be utilized in personal and vocational activities. It would also appear that contemporary psychology lends ample support to the contention that these three types of outcomes are all of them not only possible of attainment, but, under appropriate conditions, reasonably likely to be attained, in varying degree, by a large segment of the general school population.

It is the thesis of the present paper that the precise nature of the actual outcomes of mathematical instruction at any time are largely a function of the specific *purpose* envisioned by the instruction given, rather than a function of the particular mathematical content employed or the particular methodology used. As a corollary, it is submitted that the outcomes of such instruction must therefore be evaluated largely in terms of *ethical quality* and *social acceptability* of the purposes which lie behind the instruction—the climate of opinion of the specific objectives sought. This, of course, is a matter of educational philosophy and not of pedagogy. We shall attempt to support this thesis by contrasting recent trends in the teaching of mathematics in the United States with those in Nazi Germany during the decade or so before the beginning of the present war.

Before proceeding with our analysis, it should be pointed out that, since the data has been drawn largely from "primary sources", we ask the reader's indulgence with a number of passages given in the original, the meaning of which will for the most part be clear enough, while the overtones might conceivably suffer through translation, however skillful. It is also assumed that the reader is more or less familiar with the concept of the so-called *organic theory of society*, whatever significance or validity that theory may have.

Our first piece of evidence is from the pen of one writer,* who, in explaining the meaning of an "organic mathematical curriculum", asserts that the cornerstone of National Socialism is the so-called organic theory of society. In view of that, he contends, it would be a fallacy were mathematical education to strive merely for such goals as the "comprehension of abstract mathematical principles" and "mastery of mechanized manipulative processes". The ultimate outcome of instruction should instead be concerned with an integration of (1) mathematics, (2) the child, and (3) the realities of life. The desired end is to effect an organic unity that will function even as an "organic organization of society" is necessary for complete functioning. To insure the attainment of that end, the following guiding principles are proposed:

> (1) In addition to learning only abstract principles and stereotyped problems, original exercises and thought problems should be emphasized so that the pupils may achieve "abstract, productive, scientific mathematical thinking."
>
> (2) It is not sufficient to produce "mechanics" (*Techniker*) at mathematics, but "mechanicians" (*Ingenieure*) as well; e. g., pupils should not only know how to use logarithmic tables, but should also have some understanding of the relation of logarithms to exponents, to the power law, to the number concept, etc.
>
> (3) So-called applied mathematics should be made functional (*angewandte, sachgebundene Aufgaben*). In other words, beginning with concrete, familiar mathematical objects and relationships, the teacher should so organize and guide the child's learning that by successive stages the quantitative thinking in terms of everyday realities (*Sachdenken*) is transformed into abstract mathematical thinking; this generalized thinking is then in turn to be applied once more to concrete specific instances.
>
> (4) As a corollary to the last item, the instruction should strive to achieve "mathematical-mindedness", i. e., generalization not only of mathematical concepts, but also of mathematical *methods of procedure*.
>
> (5) A further aim is to develop a "sense of mathematical reality", i. e., a mathematical viewpoint or perspective concerning reality. Children should try to interpret the world of reality mathematically,—to *act* as well as think mathematically (*nach mathematischen Gesichtspunkten handeln*). Thus through mathematical instruction there is to be built up the concept of the economic man—the

*Rose, G. Der Grundriss eines organischen mathematischen Bildungsplans. *Zeit. Math. Naturw. Unt.*, 1934, 65: 1-9

keynote being utility (*Nützlichkeit*)—and the emphasis being upon thrift (*Sparsamkeit im Kauf und Verbrauch der Güter*).

(6) Mathematical instruction should be individualized. The pupil's interest, attitudes and motives must be integrated. The various capacities and personality traits must be "disciplined". This, indeed, is the chief task of mathematical instruction. (*Durch den mathematischen Unterricht soll der vom nationalen und sozialen Verantwortungsbewusstsein getragene mathematische Wirklichkeitssinn im Schüler gepflegt werden*).

(7) Finally, all economic and political questions are to be viewed mathematically.* Hence, teach percentage and graphs to "show the growth of the German nation in the last 500 years; to encourage exports; to help fight unemployment; fully one-third of all Germans are living outside of German borders; etc., etc.

In short, mathematical education is to be directed not only toward the development and realization of individual capacities and mathematical achievement, but also with a view to "setting into motion" those spiritual and intellectual forces in each pupil which will make him "an integral member of organically organized society."

Let us analyze these principles and compare them with current theory and practice at home. A superficial examination would seem to suggest that in many respects, this philosophy does not differ essentially from our own. Thus we, too, seek functional mathematical learning (transfer); we, too, want pupils to develop, through their study of mathematics, the ability to do critical thinking where quantitative relationships are involved; we, too, in the intuitive geometry of the junior high school, encourage children to pass from manipulations with concrete things to an understanding of abstract general concepts. But upon closer scrutiny, it becomes clear that there are important differences in final outcomes, despite apparent similarity of objectives; these differences are the result of significant differences of purposes and violent departures in ethical and moral concepts—a manifest perversion due to a basically different socio-economic and political philosphy. Let us compare these principles, one by one, with analogous tenets in our own philosophy of mathematical education.

(1) In recent years, mathematical instruction in American schools has striven, through the channel of problem-solving in arithmetic and algebra, and original exercises in demonstrative geometry, to develop *power* and *confidence* in problem-analysis, reflective thinking and deductive inference, avoiding stereotyped problems as well as mechanistic devices.

(2) Paralleling the second point above, current opinion in the teaching of arithmetic and algebra emphasizes the desirability and the

*"... den Raum in dem sich völkisches, deutsches Leben bewegt, im Lichte der Mathematik sehen zu lassen."

imperative need for *rationalization* of processes, the *meaningful* understanding of concepts, and the *intelligent* use of skills; we wish our pupils to know *why* as well as *how*.

(3) and (4) We, too, have been stressing as a goal the achievement of *functional* learning; of adequate *mastery;* of quantitative thinking; of mathematical competence. Writers like Betz, Hotelling, Schorling, Brownell, Wren, Reeve, and many others have repeatedly urged these very ends. Furthermore, good teaching today recognizes that transfer must be sought deliberately; it is in itself a legitimate objective of teaching. Finally, the very recent and enthusiastic interest in the use of practical applications also attests eloquently to the full realization of the significance of effective mathematical learning—mathematics that "functions".

(5) and (7) Both the significance of the role of mathematics in the modern world, as well as the realities of "social mathematics", have been important themes in mathematical instruction here at home for the past decade—actually, they both came in with the National Committee Report of 1923.

(6) The problem of adapting mathematical instruction to individual differences of capacities is equally of long standing—as a matter of fact, has given us more concern than it has abroad. Similarly, both the national reports* of the 1940 point out in no uncertain terms the importance of taking into account the special interests, individual needs, attitudes, and personalities of the pupils.

Where then, is the difference? The essential difference, I take it, lies rather in the nature of the socio-economic concepts and civic attitudes to which the child is ultimately expected to apply the outcomes of his mathematical studies. We want our young people to develop into adults capable of independent thinking; critical observers of the social scene, willing to reason things out and make their own decisions; free people with an intelligent awareness of the quantitative world and the technological civilization of which they are an integral part. We believe in appropriate mathematical education as simply one of several fundamental channels for achieving these ends. Hence the attitudes toward social, economic, vocational, technical aspects of life, insofar as reflected through mathematical material, stress independence, tolerance, self-reliance, constructive outlooks, cooperation,

*National Council of Teachers of Mathematics, 15th Yearbook, *Place of Mathematics in Secondary Education;* Progressive Education Association, *Mathematics in General Education*, D. Appleton-Century.

and wholesome adjustment. What lies behind the Nazi philosophy of education needs hardly to be commented upon here.

Another contemporary writer* enthusiastically discusses the disciplinary value of mathematical training, especially insofar as it can be of service to the Third Reich. He insists that mathematical study is "no light play",—it requires "head work, concentration, and sustained effort". Among other demands are the obligations of truth, forthrightness, and accuracy. "Mathematics, being a pure ideal, is like truth itself; it is inborn in us all. Both mathematics and truth are anti-materialistic. The spiritual values of mathematics transcend its utilitarian values, however priceless the latter may be. Both mathematics and truth demand order and discipline,—both shun and decry chaos and chance. Both are powerful, but not devoid of warmth. Cultivation of mathematics and truth lead to the greatest possible exaltation. The potentialities of mathematical education, in the hands of the inspired teacher, are still unrealized. Through mathematics we should achieve *Einheit des Menschen als Körper, Seele und Geist*". The reader may find it possible to interpret this variously as the layman's "hand, head and heart", or as the "body-mind-spirit" of the Boy Scouts, or as the "integrated whole personality" of the psychologist. But he will surely find it impossible to accept as the goal of all education *Erziehung durch* (1) *Körperpflege*, (2) *Muttersprache, und* (3) *Lehre von Blut, Boden und tätigem schöpferischen Geiste;* nor will he wish to accept the further dictum that mathematics is the cornerstone of the last of these three.

Early in the decade following the close of the first World War there appeared a book by Zühlke entitled *Politsche Mathematik*. This was followed a few years later by Weinberger's *Mathematische Volkswirtschaftslehre* and Metzner's *Staatsbürgerkunde im mathematischenaturwissenschaftlichen Unterricht.*† By 1933 a considerable body of literature on "socio-economics" had accumulated. Thus Tiedge‡ insists that all mathematical, scientific and technical resources must be pledged at the disposal of the entire people; their only justification is the common good. The greatest problem of national education, upon which the fate of the nation depends, is the improvement of the worker's status (*Erkämpfen der Arbeiter*). Because of their intimate

*Hamel, G. Die Mathematik im Dienste des dritten Reiches. *Zeit. Math. Naturw. Unt.*, 1934, 65: 10-15.

†Zuglke, P. *Politische Mathematik*. Berlin, Mittler, 1923; Weinberger, O. *Mathematische Volkswirtschaftslehre*. Leipzig, Teubner, 1930; Metzner, K. *Staatsbürgerkunde im mathematischnaturwissenschaftlichen Unterricht*. Leipzig, Quelle u. Meyer, 1931.

‡Tiedge, Ernest. Mathematik und Naturwissenschaften im Dienste der Nationalen Erziehung. *Zeit. Mathe. Naturw. Unt.*, 1933, 64: 311-320.

relation to technology and the mechanic arts (*Werktätigen Leben*), both mathematical education and science education are invaluable. "The call of the hour demands that we select materials of instruction that will benefit the good of the State (nach den Forderungen des Staatwohls). The issue is crystal-clear: *Es müssen die mathematisch-naturwissenschaftliche Erkenntnisse betont werden, die zum Bau eines festen Staatwesens und eines stark und einheitlich fühlenden und wollenden Volkes verwertet werden können.*

This new emphasis is urged even more strongly by Kerst,* who points out that whereas in former times liberal pedagogy sanctioned general all-around culture as an acceptable goal, it now behooves German education to address itself solely to the task of developing politico-economically-minded citizens (*deutsche Staatsbürger heranzubilden*). He insists further that specialization and intense cultivation of special fields should be confined to professional education, and that the schools should concentrate their efforts on equipping the youth of the land to participate actively in the life of a unified German people (*deutsche Volksgemeinschaft*). "We are not a nation of poets and thinkers, but a people of workers with hands and hearts". (*Die Aufgabe ist vielmehr den künftigen Staatsbürger vor Weltfremdheit zu bewahren, ihn auszurüsten mit dem nötigen Verständniss für die Fragen des wirklichen Lebens, für alle die Erfordernisse die dem deutschen Volke aus seinem schweren Daseinkampf ersteben ..... Der grösste Teil alles mathematischen Unterrichts ist Deutschunterricht*".)

Totalitarian *Wehrwirtschaft*, according to accepted authority, means the organization of the nation's entire economic and social life along military patterns. There are no civilians; everybody is a "soldier" of the state. The essential purpose is to transform all social relationships into something akin to military ranks, i. e., of superiors and subordinates; in this way submission to authority and economic inequalities can be rationalized more readily, since they serve military ends rather than economic purposes. In this connection, it should be pointed out that the so-called "free" or liberal professions, e. g., law, history, medicine, science, teaching, etc., since they owe their "freedom" to preoccupation with non-economic activities, are inimicable to totalitarianism. Hence it is important that they, too, be subordinated to the principles of *Wehrwirtschaft*. This presumably explains in large measure the perverted educational policies, the enunciation of a "Nazi theory of physics", and the "purification of mathematics"; these phenomena simply complete the necessary ideological background to make the acceptance of *Wehrwirtschaft* more plausible and more

*Kerst, B. Die Bedeutung der Mathematik und Physik für die deutsche Schule. *Zeit. Math. Naturw. Unt.*, 1933, 64: 149-50.

palatable. To us, however, this bit of disingeneous rationalization scarcely condones an unreasonable confounding of dispassionate scientific thinking and emotional social interpretation.

Possibly one of the contributing factors to the development of the organic theory of society was the intense interest and widespread popularity of youth movements, outdoor pageants, mass calisthenics, extensive hikes, and general participation in field sports (*Geländessport*). These leisure activities spread with phenomenal rapidity, accompanied by intriguing slogans such as *Kraft durch Freude* and the like. The universality of this interest in field activities is reflected in textbooks and periodical literature in the form of material dealing with principles of dynamics and ballistics; articles such as those by Lampe* are typical of this tendency. It would no doubt be surprising if the mathematical education of a people strongly unified in its mass-mindedness and steeped in a social philosophy of force did not sooner or later reflect its interest in the application of science and mathematics to warfare and military preparedness (*Wehrkunst*). Whether the popular interest in military applications of mathematics was a concomitant of the policy of *Wehrwirtschaft*, or whether it was deliberately fostered by Nazi leaders who (even before September, 1939) were spreading the doctrine of *Lebensraum* and the notion that the outside world was bent on annihilating them, is beside the point. The fact remains that mathematical instruction has consistently emphasized the relation of mathematics to military science, whether in the name of *Geländessport*, or *Wehrwirtschaft*, or *Staatsbürgerkunde*. Many instances of this might readily be given; we shall merely refer to one article, by Lietzmann,† as typical. Among the illustrations here given are such problems as the determination of angles and distances by indirect measurement; triangulation; determination of inaccessible heights and distances; determination of the north-south direction by means of a watch; general problems of surveying, map-making and engineering topology; study of trajectories and ballistics; the timing of bombs and projectiles to explode at a desired time; so-called "echo" problems, i. e., the determination of distances by measuring the time of the echo to return; the perspective-method of interpreting photographs of terrain taken from an airplane; and so on. The particular article referred to, although ostensibly an innocuous discussion of *Geländessport* as a natural concomitant of *Wehrwissenschaft*, clearly

*Lampe, E. Zur Mathematik der Wurfübungen im Geländessport. *Zeit. Math. Naturw. Unt.*, 1934, 65: 105-07. Die Mathematik des sportlichen Wurfes. *Zeit. Math. Naturw. Unt.*, 1927, 58:1-8.

†Lietzmann, W. Mathematischer Unterricht und Wehrwissenschaft. *Zeit. Mathe. Naturw. Unt.*, 1933, 64: 249-53.

depicts the uses of mathematics in warfare. It suggests the oft-reiterated plea in defense of *Wehrwissenschaft:* "*nicht kriegerisch, sondern soldatisch*".

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# A Note on Simple Interest

By Maximilian Philip
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1. Simple interest is paid on principal only; if interest is paid on interest the result is compound interest. The definitions imply that if a loan is repaid in instalments, equal or unequal, no part of any instalment may be considered as an interest payment if simple interest is specified, for if it were, the recipient thereof would or could earn interest thereon before the loan was liquidated and therefore earn compound interest, and the payer would be paying compound interest.

2. Simple interest is usually calculated for relatively short periods of time, usually a year or less than a year. It may be noted that if interest is at 10% per annum, the rate for 6 months is 5% if simple interest is used, but only $1.10^{\frac{1}{2}} - 1$ or $4.88\% -$ if interest is compounded semiannually.

3. Compound interest is the only logical method of computing interest. When compound interest is used, a contracted payment or set of payments may be replaced by another mode of payment by setting up an equation to show the equivalence of the given and the substituted contracts. Any date may be selected for valuing the two contracts and the result remains unchanged. When simple interest is used, the date at which valuations are made is very important, since different dates yield different results. Thus at 10% per annum \$100 at 1/1/42, \$105 at 7/1/42 and \$110 at 1/1/43 are equivalent if the comparison date is 1/1/42. But if the comparison date is 7/1/42 the value of the \$100 due at 1/1/42 is at 7/1/42 \$105, whereas the value of the \$110 due at 1/1/43 is 110/1.05 or \$104.76; if the comparison date is 1/1/43 the value of \$100 due at 1/1/42 is \$110, whereas the value of \$105 due at 7/1/42 is \$110.25.

It is therefore necessary to agree upon a comparison date whenever a debt is paid in instalments. It seems to me that the only comparison date at which an equivalence may be established when more than one payment is made is the date at which the transaction is completed, that is, the date when the final payment is made; otherwise compound interest would be involved.

4. Illustration. *A* borrowed \$1000 on 1/1/40 which was to be repaid as follows:

\$300 at 4/1/40

200 " 7/1/40

400 " 10/1/40

215 " 12/31/40 or 1/1/41

The simple interest rate is 5% per quarter or 20% per annum, which is verified as follows:

At 4/1/40 there is due \$1000+\$50 interest.

The payment of \$300 is in no part interest and after the payment is made *A* still owes \$700 and \$50 interest. At 7/1/40 the interest on \$700 is \$35, so that before any payment is made at this date *A* still owes \$700 principal and \$85 interest. After the payment of \$200 is made *A* owes \$500 principal and \$85 interest. At 10/1/40 the interest on \$500 is \$25, so that at this date *A* owes \$500 principal and \$110 interest. After the payment of \$400 is made *A* owes \$100 principal and \$110 interest. At 1/1/41 the interest on \$100 is \$5 so that at this date *A* owes \$100 principal and \$115 interest or a total of \$215 which he pays and the debt is liquidated.

More directly, taking the comparison date as the liquidation date, 1/1/41, *A* will owe \$1000 principal and \$200 interest or \$1200. At the same date he will be credited with \$300 and interest on \$300 for 3 quarterly periods or \$45, a total credit of \$345. He will also be credited with \$200 and interest for 2 quarterly periods or \$20, a total credit of \$220. He will also be credited with \$400 and interest for 1 quarterly period or a total credit of \$420. He will finally be credited with \$215 with no interest. The total credits will be \$345+220+420 +215, or \$1200, which exactly cancels his debt.

5. With the payments indicated in the problem the rate of interest is easily arrived at as follows:

At 1/1/41 *A* will owe \$1000 for 4 periods, which is equivalent to \$4000 for 1 period.

At this date he will be credited with the use of \$300 for 3 periods, equivalent to \$900 for 1 period,

with \$200 for 2 periods, equivalent to \$400 for 1 period,
with \$400 for 1 period, equivalent to \$400 for 1 period,

*A* is thus credited with the use of \$1700 for 1 period and is charged with the use of \$4000 for 1 period, a net charge of the use of \$2300 for 1 period. For this he pays \$115 more than he borrowed; that is, he pays \$115 for the use of \$2300 for 1 quarterly period and this is at the rate of 115/2300 or 5% per quarterly period.

6. If we took 1/1/40 as the comparison date, and retained the 20% rate, the value of \$300 due 4/1/40 is 300/1.05 = 285.714; the value

of \$200 due 7/1/40 is 200/1.10 = 181.818; the value of \$400 due 10/1/40 is 400/1.15 = 347.826; the value of \$215 due 1/1/41 is 215/1.20 = 179.167. At 1/1/40 the total value of all the payments is \$994.525.

That is, if interest is at 20% per annum the loan should have been \$994.515. The payment of \$300 at 4/1/40 would then have been a payment of \$285.714 principal and \$14.286 interest. Since the debt is not liquidated until 1/1/41 the recipient of the \$300 would have been able to earn interest on \$14.286 from 4/1/40 to 1/1/41. Similarly with the other payments.

The only justification that can be made for a loan of \$994.525 at 1/1/40 to be repaid as in the illustration is to consider that the loan really consisted of 4 separate loans:

285.714 payable with interest at 4/1/40

181.818 " " " " 7/1/40

347.826 " " " " 10/1/40

179.167 " " " " 1/1/41

But a loan to be paid in instalments is not broken up into a number of separate loans.

7. In the case of equal instalments my method yields a general formula which is simple.

Let the loan be $l$, payable in $n$ instalments of $p$ each at intervals of $1/k$th of a year, and the rate of interest for 1 period, $i$.

Then the value of the loan when the final payments is made is $l(1+ni)$.

The values of the instalments at the same date are

$$p[1+(n-1)i],\quad p[1+(n-2)i],\cdots\cdots p(1+i),\quad p.$$

The sum of the values of the instalments is

$$np+ip[(n-1)+(n-2)+\cdots+1+0]=pn\left[1+\frac{n-1_t}{2}\right].$$

Hence
$$l(1+ni)=pn\left[1+\frac{n-1_t}{2}\right]$$

or
$$i=\frac{p-(l/n)}{l-p(n-1/2)}$$

and the annual rate is $ki$.

Thus a \$1000 loan on 1/1/40 to be repaid by 12 equal monthly instalments of \$100 each at 1/31, 2/28, etc., would give

$$i = \frac{100 - (1000/12)}{1000 - 100(11/2)} = \frac{16\frac{2}{3}}{450} = \frac{1}{27}$$

and the annual rate is $\frac{12}{27} = \frac{4}{9} = 44\frac{4}{9}\%$.

The equation may of course be solved for $l$, $p$ or $n$.

8. When in accounting it becomes necessary to find the equitable date at which the balance in an account is to be paid, the first assumption made is that simple interest shall be used. The comparison date for this purpose should be the last date in the account. But in order that any comparison date shall be usable simple discount is used for payments after the comparison date and simple interest for payments before the comparison date. True present worth is never used because the calculation would thereby become unduly complicated.

The reason for introducing simple interest at all is not for consistency, but for convenience, to make computations easier even if the results are not quite correct.

# *Brief Notes and Comments*

*Edited by*
MARION E. STARK

16. *The Value of Mathematics in a Liberal Education.* Entirely apart from its unquestionable value as a tool of the sciences, of what value is mathematics in the liberal arts curriculum? This persistent question leads to some reflections on the nature of the relationship of mathematics to the aims of a liberal education, and some factors which enter into its effective teaching.

If it is agreed that a liberally educated person should have a knowledge of his cultural environment and of the present cultural trends, mathematics can base its claim to importance on the extent to which it has entered into our thinking. It has fundamentally influenced not only our conception of the nature of the physical universe and of the discovery of scientific laws, but also of the process of thought by which we arrive at truth.

Since science as it develops grows more mathematical in its ideas, a working knowledge of mathematics, at least through the calculus, is essential to an understanding of any description of the physical universe. An appreciation of the power as well as the limitations of the abstract, hypothetical-deductive method of thinking, and a clear idea of what proof really means, are necessary to an understanding of the meaning of truth.

Of the many factors which enter into the achievement of this desired appreciation and understanding through the study of mathematics, there are two which seem to be of particular importance. On the negative side is the danger in the tendency of the student to become so involved in the formal manipulation of the symbols that he loses sight of the significance of the fundamental principles involved, and fails to grasp the concept of mathematics as a language—the language of science and of logical thought. On the positive side, it would seem that the value of mathematics for a liberal education is a function of the extent to which it is integrated with the sciences and with philosophy. Its value varies directly with the richness of the consciously developed relationships to other fields of thought.

*Randolph-Macon Woman's College.* EVELYN WIGGIN.

# *Problem Department*

*Edited by*
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to E. P. STARKE, Rutgers University, New Brunswick, N. J.

## *SOLUTIONS*

No. 574. Proposed by *N. A. Court*, University of Oklahoma.

A sphere having its center on the line of shortest distance of the two skew lines $AB$, $CD$ meets these lines in the points $A,B$ and $C,D$ respectively. Show that $AC=BD$ and $AD=BC$.

I. Solution by *W. T. Short*, Oklahoma Baptist University.

Take the common perpendicular for the $z$-axis, and the center of the given sphere for origin. Then if the coordinates of the points $A$, $C$ are $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, the coordinates of the points $B$, $D$ are $(-x_1, -y_1, z_1)$, $(-x_2, -y_2, z_2)$. Using the distance formula we readily obtain the stated result.

II. Solution by *Frank Hawthorn*, New Rochelle, N. Y.

If $R$, $S$ are the traces on the lines $AB$, $CD$ of their line of shortest distance, we have $AR=BR$, $CS=DS$.

Let $P$, $Q$ be the projections of $A$, $B$ upon the plane through $CD$ and parallel to $AB$. We have:

$$AC^2=AP^2+PC^2, \qquad AD^2=AP^2+PD^2$$

$$BD^2=BQ^2+DQ^2, \qquad BC^2=BQ^2+CQ^2.$$

Furthermore, it follows from the construction that $AP=BQ$, and that $PCQD$ is a parallelogram, since $S$ is the midpoint of both $PQ$ and $CD$, hence $PC=DQ$, $PD=CQ$. Thus $AC=BD$, $AD=BC$.

III. Solution by *Howard Eves*, Syracuse University.

Let $O$ be the center of the sphere and $ROS$, $R$ on $AB$ and $S$ on $CD$, be the shortest distance between $AB$ and $CD$. $RS$ is then the common perpendicular to $AB$ and $CD$, and $R$ and $S$ are the midpoints of $AB$ and $CD$ respectively. Draw $C'D'$, through $R$ and having $R$ as midpoint, equal and parallel to $CD$. Draw $A'B'$, through $S$ and having $S$ as midpoint, equal and parallel to $AB$. Then $A,D',B,C',A',D,B',C$ are the vertices of a paralleopiped having rectangular lateral faces parallel to $RS$. $AC$ and $BD$ are diagonals in a pair of these opposite rectangular faces, and are therefore equal. Similarly $AD = BC$.

Also solved by *W. S. Loud, L. M. Kelly, J. S. Guérin,* and *P. D. Thomas.*

Thomas points out that the common perpendicular considered is both a bialtitude and a bimedian of the tetrahedron $ABCD$ and he refers to Art. 202, p. 61 of Court's *Modern Pure Solid Geometry,* (Macmillan, 1935).

No. 575. Proposed by *Victor Thébault,* Tennie, Sarthe, France.

The four pairs of points $A,A'$; $B,B'$; $C,C'$; $D,D'$ are marked on four parallel non coplanar lines. The points $P$, $Q$, $R$, $S$ divide the segments $AA'$, $BB'$, $CC'$, $DD'$ in the same ratio $k$; and the points $P'$, $Q'$, $R'$, $S'$ are the symmetrics of $P$, $Q$, $R$, $S$ with respect to the midpoints of those segments. Show that the sum of the volumes of the two tetrahedrons $PQRS$, $P'Q'R'S'$ remains constant when the value of $k$ varies. Also, determine the value of $k$ for which the points $P$, $Q$, $R$, $S$, or the points $P'$, $Q'$, $R'$, $S'$ are coplanar

I. Solution by *J. S. Guérin,* Catholic University of America, Washington, D. C.

To fix the ideas, let $AA'$ lie within the volume $BCDB'C'D'$ so that $A$, $A'$ are exterior to this volume.

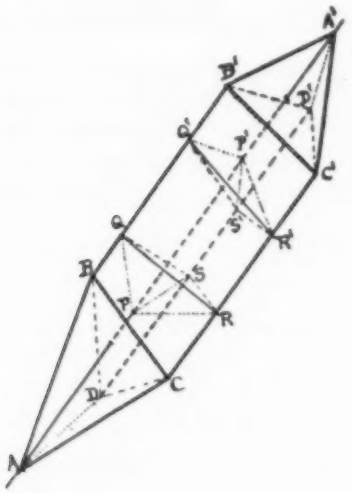
The volume of a truncated triangular prism is equal to the product of the right section by the arithmetic mean of the three edges, hence, by proportions, we have:

$$\text{(1)} \qquad \frac{\text{vol. } ABCPQR}{\text{vol. } ABCA'B'C'} = \frac{k}{k+1}$$

$$\text{(2)} \qquad \frac{\text{vol. } ABDPQS}{\text{vol. } ABDA'B'D'} = \frac{k}{k+1}$$

$$\text{(3)} \qquad \frac{\text{vol. } ACDPRS}{\text{vol } ACDA'C'D'} = \frac{k}{k+1}$$

$$\text{(4)} \qquad \frac{\text{vol. } BCDQRS}{\text{vol. } BCDB'C'D'} = \frac{k}{k+1}.$$




Comparing (1), (2), (3) and transforming, we have

$$\text{(5)} \qquad \frac{\text{vol. } BCDQRS - \text{vol. } PQRS + V}{V + V_0 + V'} = \frac{k}{k+1},$$

where, $V = \text{vol. } ABCD$

$V_0 = \text{vol. } BCDB'C'D' \qquad (V > V')$

$V' = \text{vol. } A'B'C'D'.$

Again, comparing (4) and (5) and transforming, we have

$$\frac{V - \text{vol. } PQRS}{V + V'} = \frac{k}{k+1}.$$

Hence, $$\text{vol. } PQRS = \frac{V - kV'}{k+1}.$$

Similarly, $$\text{vol. } P'Q'R'S' = -\frac{(V' - kV)}{k+1}.$$

Adding, $$\text{vol. } PQRS + \text{vol. } P'Q'R'S' = V - V'.$$

If $P$, $Q$, $R$, $S$, are coplanar, $$k = \frac{V}{V'}.$$

If $P'$, $Q'$, $R'$, $S'$ are coplanar, $k=\dfrac{V'}{V}$.

II. Solution by *Howard Eves*, Syracuse University.

Select a three dimensional rectangular coordinate system having the $z$-axis parallel to the four given lines. We may then designate the points $A,A'$; $B,B'$; $C,C'$; $D,D'$ by the sets of coordinates $(a_1,a_2,p)$, $(a_1,a_2,p')$; $(b_1,b_2,q)$, $(b_1,b_2,q')$; $(c_1,c_2,r)$, $(c_1,c_2,r')$; $(d_1,d_2,s)$, $(d_1,d_2,s')$ respectively. Then the coordinates of $P,Q,R,S$, are

$$\left(a_1,\ a_2,\ \frac{p+kp'}{1+k}\right),\qquad \left(b_1,\ b_2,\ \frac{q+kq'}{1+k}\right),$$

$$\left(c_1,\ c_2,\ \frac{r+kr'}{1+k}\right),\qquad \left(d_1,\ d_2,\ \frac{s+ks'}{1+k}\right),$$

and the coordinates of $P',Q',R',S'$ are

$$\left(a_1,\ a_2,\ \frac{kp+p'}{1+k}\right),\qquad \left(b_1,\ b_2,\ \frac{kq+q'}{1+k}\right),$$

$$\left(c_1,\ c_2,\ \frac{kr+r'}{1+k}\right),\qquad \left(d_1,\ d_2,\ \frac{ks+s'}{1+k}\right).$$

We therefore have (designating determinants by only the first row)

$$\begin{aligned}
&6(\text{vol. } PQRS+\text{vol. } P'Q'R'S')\\
&=\left|a_1,\ a_2,\ \frac{p+kp'}{1+k},\ 1\right|+\left|a_1,\ a_2,\ \frac{kp+p'}{1+k},\ 1\right|\\
&=|a_1,\ a_2,\ p+p',\ 1|\\
&=|a_1,\ a_2,\ p,\ 1|+|a_1,\ a_2,\ p',\ 1|\\
&=6(\text{vol. } ABCD+\text{vol. } A'B'C'D').
\end{aligned}$$

This establishes the first part of the problem.

To determine the value of $k$ for which $P,Q,R,S$ are coplanar we note that, in this case,

$$\left|a_1,\ a_2,\ \frac{p+kp'}{1+k},\ 1\right|=0,\qquad \text{whence}$$

$$k=-|a_1,\ a_2,p,\ 1|/|a_1,\ a_2,\ p',\ 1|=-\text{vol. } ABCD/\text{vol. } A'B'C'D'.$$

Similarly $\qquad k=-\text{vol. } A'B'C'D'/\text{vol. } ABCD$

makes $P',Q',R',S'$ coplanar.

It is important to notice that in the above we are employing *signed* volumes. Two tetrahedra $ABCD$, $A'B'C'D'$ will have volumes of the same sign if the oblique systems $A-BCD$ and $A'-B'C'D'$ are of the same type. Otherwise the volumes will be of unlike sign.

No. 579.* Proposed by *Walter B. Clarke*, San Jose, Calif.

The median of the triangle $ABC$ issued from $A$ meets the circumcircle again in $D$, and the tangents at $A$ and $D$ to that circle meet $BC$ in $E$ and $F$. Show that $AE=FD$.

I. Solution by *L. M. Kelly*, Coast Guard Academy, New London, Connecticut.

Let $O$ be the circumcenter of $ABC$ and $M$ the mid-point of $BC$. The two circles having $OE$ and $OF$ for diameters pass, respectively, through the two pairs of points $A$, $M$ and $D$, $M$. Taking into account the two cyclic quadrilaterals $AOME$, $DOMF$, we have:

$$\measuredangle AOE=\measuredangle AME=\measuredangle FMD=\measuredangle FOD,$$

hence the two right triangles $AOE$, $DOF$ are congruent, and $AE=DF$.

II. Solution by *H. E. Fettis*, Dayton, Ohio.

If we consider each of the points $A$, $D$ as a pair of coincident vertices of a complete quadrangle $(Q)$ inscribed in the given circle $(O)$, then the lines $AE$ and $DF$ are a pair of opposite sides of $(Q)$ and $AD$ a second pair of coincident opposite sides of $(Q)$.

By Desargues' theorem $E$, $F$ and $B$, $C$ are thus two pairs of points of an involution of which the trace $M$ of $AD$ on $BC$ is a double point. But $M$ is the mid-point of $BC$, by assumption, hence $M$ is also the mid-point of $EF$, and we have therefore

$$BE=CF, \qquad BF=CE.$$

Now considering the powers of the points $E$, $F$ for $(O)$ we have:

$$EA^2=EB\cdot EC, \quad FD^2=FC\cdot FB,$$

hence

$$EA=FD.$$

Also solved by *Frederick M. Carpenter*, *J. S. Guérin*, *Howard Eves*, and *P. D. Thomas*.

Eves solved the problem in five different ways, including a projective generalization which involves a pencil of conics. He also

*October, 1944. Due to an oversight, there are two problems each numbered 579. 580. 591; another set in the November issue.

pointed out the analogy between this problem and the Butterfly Theorem in the *American Mathematical Monthly*, February, 1944, p. 91, Q. E571.

No. 580.* Proposed by *Howard Grossman*, New York City.

Find $\lim_{n\to\infty} \cos\frac{\pi}{3}\cos\frac{\pi}{4}\cos\frac{\pi}{5}\cdots\cos\frac{\pi}{n}$.

(See Kasner's *Mathematics and the Imagination*, p. 311, for an interesting geometric interpretation.)

EDITORIAL NOTE. Since the infiite product converges slowly, we may proceed as follows. Note that, for $n>3$,

$$1>\cos \pi/n>(1-\pi^2/2n^2)>0,$$

which follows from the familiar Taylor's expansion of the cosine. Hence

$$(1) \qquad 1>\prod_{n=a}^{\infty}\cos \pi/n>\prod_{n=a}^{\infty}(1-\pi^2/2n^2).$$

Following a suggestion by *Harold E. Bowie*, we may use the relation

$$\sin z=z\prod_{n=1}^{\infty}(1-z^2/n^2\pi^2)$$

valid for $z\neq k\pi$. (See Pierpont, *Functions of a Complex Variable*, p. 281.) Putting $z=\pi^2/\sqrt{2}$, we have $\pi^2/2n^2=z^2/n^2\pi^2$, and

$$(2) \qquad \prod_{n=1}^{\infty}\left(1-\frac{\frac{1}{2}\pi^4}{n^2\pi^2}\right)=\frac{\sin(\pi^2/\sqrt{2})}{\pi^2/\sqrt{2}}=.091836.$$

We use tables to compute

$$(3) \qquad \prod_{n=3}^{25}\cos \pi/n=.13950, \quad \prod_{n=1}^{25}(1-\pi^2/2n^2)=.11179.$$

Thus we have, using (2),

$$\prod_{n=26}^{\infty}(1-\pi^2/2n^2)=.82380,$$

and, after multiplying (1) through by the first result in (3),

$$.1395>\prod_{n=3}^{\infty}\cos \pi/n>.1149.$$

For $n>25$, $\cos \pi/n$, and $1-\pi^2/2n^2$ differ by less than .00001. Thus the lower of the bounds given above is much closer to the correct value, which can be given to three decimal places as .115.

*November, 1944.

No. 582. Proposed by *E. P. Starke*, Rutgers University.

Find the smallest three-digit number $N$ such that the three numbers obtained from $N$ by cyclic permutations of its digits are in arithmetic progression.

Solution by *Aida Kalish*, Smith College.

Let such a number be represented as $abc$, so that the other terms of the arithmetic progression are $bca$ and $cab$. For the three numbers

$$100a+10b+c, \qquad 100b+10c+a, \qquad 100c+10a+b$$

to be in arithmetic progression we must have

$$2(100b+10c+a)=(100a+10b+c)+(100c+10a+b)$$

which reduces to $$7b-4a=3c.$$

If $a=1$, there is only one non-trivial value for $b$ which will make the left side a multiple of 3. That is $b=4$ which gives $c=8$. Thus 148 is the smallest possible required number. Other such numbers are 259, 851 and 962, and the unacceptable values 037 and 740. Oddly enough, all resulting progressions have common difference of 333.

Also solved by *J. F. Arena*, *Harold E. Bowie*, *Howard Eves*, *J. S. Guérin*, *B. A. Hausmann*, *Emil D. Schell*, *Lloyd Tennenbaum*, and *P. D. Thomas*. Note a similar problem in the *American Mathematical Monthly*, February, 1945, p. 95, in which geometric progression is required.

No. 584. Proposed by *Howard Eves*, Syracuse University.

Show that $$\sum_{j=0}^{n}\binom{n}{j}(-1)^j/(j+1)=1/(n+1).$$

I. Solution by *Emil D. Schell*, Arlington, Virginia.

$$\begin{aligned}0&=(1-1)^{n+1}\\&=1-\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\\&=1-(n+1)\sum_{j=0}^{n}\binom{n}{j}(-1)^j/(j+1),\end{aligned}$$

from which the desired formula is obvious.

II. Solution by *Harry M. Gehman*, University of Buffalo.

Given the identity

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^j x^j=(1-x)^n,$$

if we multiply each side by $dx$ and integrate from 0 to 1, we obtain

$$\int_0^1 \sum_{j=0}^{n} \binom{n}{j} (-1)^j x^{j+1}/(j+1) = -(1-x)^{n+1}/(n+1) \Big]_0^1 = 1/(n+1).$$

III. Solution by the *Proposer*.

Let
$$\frac{n!}{x(x+1)\cdots(x+n)} = \frac{A_0}{x} + \frac{A_1}{x+1} + \cdots + \frac{A_j}{x+j} + \cdots + \frac{A_n}{x+n}.$$

Clearing of fractions and setting $x = -j$ we find

$$n! = A_j(-j)(-j+1)\cdots(-1)(+1)\cdots(-j+n)$$
$$= A_j(-1)^j\, j!(n-j)!$$

Therefore $A_j = n!(-1)^j/j!(n-j)! = \binom{n}{j}(-1)^j$.

Thus we have

$$\frac{n!}{x(x+1)\cdots(x+n)} = \sum_{j=0}^{n} \frac{\binom{n}{j}(-1)^j}{x+j}.$$

Setting $x = 1$ we have the desired relation.

Also solved by *Henry E. Fettis*.

## *PROPOSALS*

No. 612. Proposed by *Howard Eves*, Syracuse University.

Towards the bottom of p. 268 of Cajori's *A History of Mathematics*, 2nd Ed., 1919, we find the following statements:

"Napoleon proposed to the French mathematicians the problem, to divide the circumference of a circle into four equal parts by the compasses only. Mascheroni does this by applying the radius three times to the circumference; he obtains the arcs $AB$, $BC$, $CD$; then $AD$ is a diameter; the rest is obvious."

Complete the "obvious" part of the construction.

No. 613. Proposed by *V. Thébault*, Tennie, Sarthe, France.

In an isosceles tetrahedron the three lines joining the center of an escribed sphere to the vertices of the corresponding face are mutually perpendicular; and conversely.

No. 614. Proposed by *Merton T. Goodrich*, Keene, N. H.

Find all right triangles such that the sides are integers and the perimeter is equal to the area.

No. 615. Proposed by *W. E. Byrne*, Virginia Military Academy.

Consider the real roots in the interval $-\pi \leqq \phi \leqq \pi$ of the equation

$$f(\phi) = \omega^2 b \sin 2\phi - 3(g \sin \phi - \omega^2 a \cos \phi) = 0,$$

where $\omega$, $a$, $b$, $g$ are positive constants. Show there may be two or four distinct roots, or three distinct roots of which one is double. What relations must hold among the constants to give a double root?

No. 616. Proposed by *N. A. Court*, University of Oklahoma.

Find the locus of a point which moves so that the lines joining it to the vertices of a tetrahedron form an orthocentric group. Discuss special cases.

No. 617. Proposed by *Howard Eves*, Syracuse University.

If one side of a given complete quadrilateral is parallel to the Euler line of the triangle formed by the remaining three sides, the same is true for every side of the quadrilateral.

No. 618. Proposed by *V. Thébault*, Tennie, Sarthe, France.

In the system of numeration having base 11, find a six-digit square of the form $a\dot{b}cabc$.

# *Bibliography and Reviews*

*Edited by*
H. A. SIMMONS and P. K. SMITH

***Fundamentals of Algebra.*** By Joseph A. Nyberg. American Book Company, New York, 1944. v+336 pages. $1.40.

This book includes material for a year course in beginning algebra. It treats *graphs, formulas, linear equations, exponents, factoring, quadratic equations, and proportion and variation*[1] and the last chapter, which is on *trigonometric ratios*, adds much to the perspective of students. The text emphasizes duly the importance of *drill* in the various manipulations, especially with fractions; and about 15 pages of tests should help the student to measure his progress from time to time, and to eliminate weakness in various types of detail.

There is an average of approximately one graph to every four pages of the book. This indicates that the author considered graphs as aids to clearness, or ease, or both of these, in thinking. We agree with him on this point. We also approve of his idea of starting with the simplest types of statistical graphs and gradually working up to the graphing of quadratics in two variables.

Another interesting feature of this text is its tendency to depict the role of mathematics in every-day life. For instance, we find paragraphs on such topics as "Formulas for the Farmer", "Who uses mathematics"? "Parentheses Are Something like a Basket", "Equations Aid the Miller", "Dependence along the Highway", "Scientists Must Know How to Factor", "Fractional Equations on the River", "The Building Trades Use Mathematics". Numerous pictures of life situations, with accompanying stories, should strengthen the subjects' appeal to the student.

The text is unquestionably a very good one for first year algebra. Further, students who have had algebra and have been away from it for several years will also find this a good book to use in reviewing the subject.

*Northwestern University.* H. A. SIMMONS.

***Plane and Spherical Trigonometry.*** By Howard P. Doole. Thomas Y. Crowell Company, 1944. viii+183 pages. $1.75.

The brief text (167 pages, 86 sections, 1028 problems and exercises, 86 figures) under review, is divided into chapters (153 pages, 78, 981, 82) numbered I to VII and appendices (14, 8, 47, 4) lettered A to C. These bear appropriate captions as indicated below: I. *Introduction and Definitions* (32, 16, 253, 26); II. *Identities and Equations Involving One Angle Only* (12, 6, 102, 1); III. *Functions of Multiple Angles* (22, 8, 231, 9); V. *Solution of Triangles* (30, 15, 136, 19); VI. *Line Values and Graphs of Trigonometric Functions* (10, 6, 41, 10); VII. *Spherical Trigonometry* (32, 20, 121, 16); A. *The Mil* (3, 1, 22, 0); B. *The Slide Rule* (6, 5, 13, 2); C. *Vectors* (5, 2, 12, 2). Four place tables of logarithms and natural functions, an index and answers to odd-numbered problems occupy 2 and 1, 5 and 5 pages respectively.

The author's announced purpose "to insure all students an opportunity to acquire solid foundation in the trigonometry needed for any future courses in mathematics" is an ambitious one which has been attained fairly well, in the reviewer's opinion.

Brevity has been gained by such devices as including the work on radian measure with the initial discussion of angles, placing the work on the solution of right as well as oblique plane (spherical) triangles in a single chapter, omitting the discussion of the functions of the *sum of an angle and all integral multiples of* $\frac{1}{2}\pi$ as a separate topic, omitting or specializing proofs of some of the formulas which are used.

The "inverse trigonometric functions are presented" in the first chapter "and then used as often as possible throughout the course." The reader experiences difficulty sometimes with equalities (p. 22) because these functions are multiple valued, and later with the proofs of identities (pp. 62-63) either because the values of the variables must be limited to fit the stated restriction to principal values (Ex. 3, 21-26), or because the identity is not true under this restriction (Ex. 8, 15, 16). Again, exercises (p. 32) require interpolation before it has been explained. More generally, informal language such as "to rotate an infinite number of times" (p. 22), "the $2\pi$ position" (p. 24), "and only these eight are all that" (p. 35), "logical or approximate algebraic simplification" (p. 37), "two angles under 360" (p. 42), "and inverted" (p. 116), add only rarely to the clarity of meaning.

Technical features of the book include a pleasing format, well placed material on the (5″x6¾″) pages, graded sets of well diversified exercises, and attractive diagrams. and graphs.

*The University of Omaha.* JAMES M. EARL.

*Theorie und Andwendung der Laplace-Transformation.* By Gustav Doetsch. Die Grundlehren der Mathematischen Wissenschaften in Einzeldanstellungen mit besonderer Berüchsichtigung der Anwendungsgebiete. Band XLVII. Published and distributed in the public interest by authority of the U. S. Alien Property Custodian under license No. A-188. Dover Publications, 1943. xiv+499 pages.

This book was first published in Berlin, by Springer, in 1937. It was reviewed by R. V. Churchill in the Bulletin of the American Mathematical Society, Vol. 44 (1938) pp. 476-477. In the preface to the present American edition, the editor states:

"Gustav Doetsch's world renowned monograph on the theory and practical applications of the Laplace Transformation is generally recognized as the most useful and rigorous exposition of an operational method which has become increasingly important for wartime research problems of the mathematician and physicist.

"The publishers are therefore gratified to make this book available to the American scientist at less than one-fourth of the former $14.50 price.

"For greater convenience of American users corrections have been made to all errors in the first edition noted by the editor. The German-English glossary of technical terms peculiar to this volume has been compiled by Dr. Sirgei Feitelberg, physicist for Mt. Sinai Hospital."

The book is entirely in German except for the glossary just mentioned. In 1939, the book was translated and abridged by W. A. Mersman, instructor in mathematics and physics at the University of California, and a typewritten copy was reproduced and made available through the department of mechanical engineering and WPA Project No. 665-08-3-223. This translation, 87 pages in length, is mainly a statement of theorems and results, with proofs omitted, and with the applications generally indicated by title only

If a function $F(t)$ satisfies certain conditions ("lies in the $L$-space") a function $f(s)$ is defined by the relation

$$(1) \qquad f(s) = \int_0^\infty e^{-st}F(t)dt \equiv L[F(t)],$$

and $f(s)$ is called the Laplace transform of $F(t)$. The function $f(s)$ is referred to as a point in the 1-space. The transformation (1) is from a point in the $L$-space to a point in the 1-space. It is a linear transformation, possessing properties of fundamental importance, these being expressed by nine formulas, which, for convenience are omitted.

By use of these formulas, the solving of linear differential equations, difference equations, or integral equations, is made to depend on the solving of simpler equations and of finding the inverse Laplace transforms of functions.

For example, to solve the differential-difference system

$$Y'(t)-a\,Y(t-1)=b \text{ when } t>0,$$
$$=0 \text{ when } t<0,$$
$$Y(t)=0 \text{ when } t\leq 0,$$

we take the Laplace transform, and obtain the simpler equation for $y(s)$,

$$s\,y(s)-ae^{-s}y(s)=b/s.$$

The differential-difference equation in $Y(t)$ is thus replaced by an algebraic equation in its transform $y(s)$. We solve for $y(s)$ and obtain

$$y(s)=\frac{b}{s^2}\left[1-\frac{a}{s}e^{-s}\right]^{-1}$$
$$=b\left[\frac{1}{s^2}+\sum_{n=1}^{\infty}a^n\frac{1}{s^{n+2}}e^{-ns}\right].$$

We then find the inverse Laplace transform

$$Y(t)=b\left[t+\sum_{n=1}^{\infty}\frac{a^n}{(n+1)!}(t-n)^{n+1}\right],$$

where $(t-n)^{n+1}$ is to be replaced by zero when $t<n$.

Similarly systems of linear ordinary differential equations may be replaced by systems of algebraic equations, and a linear partial differential equation may be replaced by a linear ordinary differential equation. The solving for the transform may be simple, but the finding of its inverse may be difficult.

Doetsch's book contains a careful discussion of the conditions under which formulas hold, and gives a systematic presentation of the general theory of the Laplace-Transform. An interesting and important part of the theory is that pertaining to the range of $s$ in the complex plane. Starting with a real function $F(t)$, the transform defines a function of a complex variable $s$, related to the Fourier-transform of a function.

The reader will find in R. V. Churchill's recent book "Modern Operational Mathematics in Engineering," McGraw-Hill, 1944, a presentation in English of portions of the material presented in Doetsch, together with extensive illustrations of the use of the method in solving problems in mathematical physics and engineering. This is a valuable companion book for the study of Doetsch's more complete treatise, especially for persons interested in applied mathematics.

*Northwestern University.* E. J. MOULTON.

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